Symmetric Matrices

(Remark) eigenvalues can be complex (even if A is a real matrix)

For ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightsquigarrow |A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$

$|A - \lambda I| = 0 \Rightarrow \lambda^2 + 1 = 0$, or $\lambda = \pm i$

note: $i^2 = -1$ or $i = \sqrt{-1}$

What happens when $A \in \mathbb{R}^{2\times 2}$ is symmetric

Let $A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \rightsquigarrow |A - \lambda I| = \begin{vmatrix} a-\lambda & c \\ c & b-\lambda \end{vmatrix}$

$|A - \lambda I| = 0 \Rightarrow (a-\lambda)(b-\lambda) - c^2 = 0$

$ab - (a+b)\lambda + \lambda^2 - c^2 = 0$

$$\Rightarrow \lambda = \frac{(a+b) \pm \sqrt{(a+b)^2 + 4(c^2 - ab)}}{2}$$

$$= \frac{(a+b) \pm \sqrt{(a-b)^2 + 4c^2}}{2}$$

non negative

$\boxed{\text{Thm}}$ Let $A \in \mathbb{R}^{n\times n}$ with $A = A^T$

Then A has real eigenvalues

EXAMPLE Let $A = \begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}$

Let's find the eigenvalues and eigenvectors of A

eigenvalues $A - \lambda I = \begin{pmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{pmatrix}$ $|A - \lambda I| = 0 \Rightarrow$ $(2-\lambda)(1-\lambda) - 2 = 0$

$2 - \lambda - 2\lambda + \lambda^2 - 2 = 0$

$\lambda^2 - 3\lambda = 0$

or $\lambda = 0$ or 3

eigenvectors $\lambda = 0$ we want $\vec{x}$ s.t. $A\vec{x} = \vec{0}$ $\vec{x} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$ is an eigenvector

now consider $\vec{y} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$ which is perpendicular to $\vec{x}$

Then we check $\underbrace{\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}}_{\vec{y}} = \begin{pmatrix} 3\sqrt{2} \\ 3 \end{pmatrix} = \underbrace{3}_{\lambda} \underbrace{\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}}_{\vec{y}}$ $\vec{y}$ is an eigenvector!

$\boxed{\text{Thm}}$ The eigenvectors of symmetric matrices (w/ distinct eigenvalues) are always orthogonal.

Eigenvalue | Eigenvector Remarks

Eigenvalues of $A \in \mathbb{R}^{n \times n}$

- If λ is a root of $\det(A - \lambda I) = 0$ then $\bar{\lambda}$ is also a root
  conjugate

note:

If $\lambda = a + ib$
then
$\bar{\lambda} = a - ib$

General A: λ is an eigenvalue $\Leftrightarrow$ $\bar{\lambda}$ is also an eigenvalue

Symmetric A ($A = A^T$): all eigenvalues are real, always

Diagonalizability

General A: A might not be diagonalizable if it has repeated (non-distinct) eigenvalue

example: Suppose $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ both eigenvalues are 1
but only $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector

$A = A^T$: A is always diagonalizable, even if eigenvalues repeat

Eigenvector Remarks (Contd....)

General A Eigenvectors for distinct eigenvalues are linearly independent (See #6.2)

When $A = A^T$

(Spectral Theorem) Every symmetric matrix has the factorization

$A = Q\Lambda Q^T$ with real eigenvalues in Λ and orthonormal eigenvectors in Q

Symmetric Diagonalization $A = Q\Lambda Q^T = Q\Lambda Q^{-1}$ with $Q^{-1} = Q^T$

Symmetric A $\rightsquigarrow$ eigenvectors that form an orthonormal basis of $\mathbb{R}^n$

Example $\quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 2 \end{pmatrix}$

Eigenvalues $\quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & \sqrt{3} \\ 0 & \sqrt{3} & 2-\lambda \end{vmatrix}$

$$|A - \lambda I| = 0 \Rightarrow (1-\lambda)(-\lambda)(2-\lambda) - 3 = 0$$

$$(1-\lambda)(\lambda^2 - 2\lambda - 3) = 0$$

$$(1-\lambda)(\lambda-3)(\lambda+1) = 0$$

or, $\quad \lambda = 1, 3, -1$

Eigenvectors $\quad \underline{\lambda = 1}$ ...... we get $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\underline{\lambda = -1}$ ...... $\vec{x}_2 = \begin{pmatrix} 0 \\ -\sqrt{3} \\ 1 \end{pmatrix}$ or $\vec{x}_2 = \frac{1}{2}\begin{pmatrix} 0 \\ -\sqrt{3} \\ 1 \end{pmatrix}$

(normalized to unit length)

$\underline{\lambda = 3}$ ...... $\vec{x}_3 = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ \sqrt{3} \end{pmatrix}$

Diagonalization

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix}}_{Q^{-1} = Q^T}$$

<u>Example</u> Compute e^{At} for $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 2 \end{pmatrix}$

Recall,

$$e^{At} := I + At + \frac{t^2}{2} A^2 + \ldots\ldots + \frac{t^j}{j!} A^j$$

and,

$$A^j = S \Lambda^j S^{-1} = Q \Lambda^j Q^T \quad \text{where } Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\sqrt{3}/2 & 1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Then,

$$e^{At} = \sum_{j=1}^{\infty} Q \begin{pmatrix} \frac{t^j}{j!} & 0 & 0 \\ 0 & \frac{(-t)^j}{j!} & 0 \\ 0 & 0 & \frac{(3t)^j}{j!} \end{pmatrix} Q^T$$

$$= Q \begin{pmatrix} e^{t} & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{3t} \end{pmatrix} Q^T$$