EIGENVALUES AND EIGENVECTORS

REVIEW

Def^n Suppose $A \in \mathbb{C}^{n \times n}$, then $\vec{x} \neq \vec{0}$, $\vec{x} \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$ (scalar) are eigenvectors and eigenvalues respectively if

$$A\vec{x} = \lambda\vec{x}$$

THEOREM (Eigenvalues of a matrix)

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

(Proof sketch) Let λ be an eigenvalue of A

$\Leftrightarrow$ there exists $\vec{x} \neq 0$ so that $A\vec{x} = \lambda\vec{x}$

$\Leftrightarrow$ — " — $A\vec{x} - \lambda\vec{x} = \vec{0}$

$\Leftrightarrow$ — " — $A\vec{x} - \lambda I_n\vec{x} = \vec{0}$

$\Leftrightarrow$ — " — $(A - \lambda I_n)\vec{x} = \vec{0}$

$\Leftrightarrow$ $A - \lambda I_n$ is singular

$\Leftrightarrow$ $\det(A - \lambda I_n) = 0$

- the polynomial $p_A(\lambda) = \det(A - \lambda I_n)$ is the characteristic polynomial of A
- when A is $n \times n$, $p_A(\lambda)$ has degree n. hence A has n eigenvalues (note: repeats, complex values possible)

SUMMARY - THE EIGENVALUE PROBLEM

REVIEW

To solve the eigenvalue problem for an $n \times n$ matrix, follow these steps:

Step 1

Compute the determinant of $A - \lambda I$. (this is a polynomial of degree n in λ)

Step 2

Find the roots of this polynomial by solving $\det(A - \lambda I) = 0$.

The n roots are the n eigenvalues of A

Step 3

For each eigenvalue λ_i, solve $(A - \lambda_i I)\vec{x}_i = \vec{0}$ to find an eigenvector $\vec{x}_i$

(for $i = 1, 2, \ldots, n$)

DIAGONALIZING A MATRIX

REVIEW

Suppose that $A \in \mathbb{R}^{n\times n}$ has n eigenvectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ that are linearly independent

Let $$X = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$$ is full rank (invertible)

$$\Rightarrow \quad AX = \begin{pmatrix} | & | & & | \\ \lambda_1 \vec{x}_1 & \lambda_2 \vec{x}_2 & \cdots & \lambda_N \vec{x}_n \\ | & | & & | \end{pmatrix} = X \Lambda$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$ (diagonal matrix)

$$\Rightarrow \quad X^{-1} A X = \Lambda$$

here A "looks diagonal"

or

when viewed "through eigenvectors"

$$A = X \Lambda X^{-1}$$

Thm Eigenvectors that correspond to different eigenvalues are linearly independent.

REVIEW

Proof Suppose $A\vec{x}_1 = \lambda_1 \vec{x}_1$, and $A\vec{y} = \mu \vec{y}$ with $\lambda_1 \neq \mu$.

$\vdots$

$A\vec{x}_m = \lambda_m \vec{x}_m$

(distinct/different eigenvalues)

Then, $\vec{x}_1 \in N(A - \lambda_1 I)$; so $\operatorname{span}\{\vec{x}_1\} \subseteq N(A - \lambda_1 I)$

$\operatorname{span}\{\vec{x}_1, \dots, \vec{x}_m\}$

However (from previous slide) $(A - \lambda_1 I)\vec{y} = \underbrace{(\mu - \lambda)}_{\neq 0}\vec{y}$ $\Rightarrow$ $\vec{y} \notin N(A - \lambda_1 I)$

$\Rightarrow$ $y \notin \operatorname{span}\{\vec{x}_1\}$

$\operatorname{span}\{\vec{x}_1, \dots, \vec{x}_m\}$

Therefore, if $A \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, its eigenvector matrix

$$X = \begin{pmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{pmatrix}$$

will be invertible.

DIFFERENTIAL EQUATIONS

(REVIEW)

$$\frac{dx}{dt} = x \iff x(t) = e^t$$

$$\frac{dx}{dt} = cx \iff x(t) = e^{ct} \quad \text{where } c \in \mathbb{R}$$

* under the assumption $x(0)=1$

In general, we have

$$\frac{dx}{dt} = x \iff x(t) = x(0)\, e^t$$

$$\frac{dx}{dt} = cx \iff x(t) = x(0)\, e^{ct}$$

Systems of Differential Equations

Consider

$$\left.\begin{aligned} \frac{dx}{dt} &= ax + by \\[2ex] \frac{dy}{dt} &= cx + dy \end{aligned}\right\}$$

Coupled set of differential equations

These equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\vec{x}} \iff \frac{d\vec{x}}{dt} = A\vec{x}$$

Note: $\frac{d\vec{x}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix}$

$$A\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} a\cdot\frac{dx}{dt} + b\cdot\frac{dy}{dt} \\ c\,\frac{dx}{dt} + d\cdot\frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}(ax+by) \\ \frac{d}{dt}(cx+dy) \end{pmatrix}$$

$$= \frac{d}{dt} A \begin{pmatrix} x \\ y \end{pmatrix}$$

When A is diagonalizable

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

$$\Big\downarrow \text{ diagonalizable } A$$

$$\frac{d\vec{x}}{dt} = (V \Lambda V^{-1})\vec{x} \quad \text{where } V = \begin{pmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{pmatrix}$$

eigenvectors (linearly independent)

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & \lambda_n \end{pmatrix}$$

(diagonal matrix)

$$\Downarrow$$

$$V^{-1}\frac{d\vec{x}}{dt} = \Lambda (V^{-1}\vec{x})$$

(multiplying both sides by V^{-1})

$$\Downarrow$$

$$\frac{d}{dt}(V^{-1}\vec{x}) = \Lambda (V^{-1}\vec{x})$$

Let $\vec{y} = V^{-1}\vec{x}$

$$\Rightarrow \frac{d\vec{y}}{dt} = \begin{pmatrix} \lambda_1 y_1 \\ \lambda_2 y_2 \\ \vdots \\ \lambda_n y_n \end{pmatrix} \quad \text{-or-} \quad \begin{aligned} \frac{dy_1}{dt} &= \lambda_1 y_1 \\ \frac{dy_2}{dt} &= \lambda_2 y_2 \\ &\vdots \end{aligned} \Longrightarrow \begin{aligned} \underline{y_1(t) = y_1(0)e^{\lambda_1 t}} \\ \underline{y_2(t) = y_2(0)e^{\lambda_2 t}} \end{aligned}$$

If $y_1(0) = y_2(0) = 1$,

$$\vec{y} = \begin{pmatrix} e^{\lambda_1 t} \\ e^{\lambda_2 t} \end{pmatrix}$$

Example: Solve $\dfrac{d\vec{u}}{dt} = \underbrace{\begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}}_{A} \vec{u}$

Let's diagonalize A!

eigenvalues $\quad |A-\lambda I| = \begin{vmatrix} -2-\lambda & 3 \\ 2 & -3-\lambda \end{vmatrix} = (\lambda+2)(\lambda+3)-6$

$|A-\lambda I| = 0 \Rightarrow \lambda^2 + 5\lambda = 0$ or $\lambda = 0,\ \lambda = -5$

eigenvectors $\quad \lambda = 0$ (nullspace of A) $\quad \vec{v}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda = -5$ solve $(A+5I)\vec{x} = \vec{0}$ or $\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\vec{x} = \vec{0}$ or $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

A has distinct eigenvalues (A is diagonalizable) $\therefore$ $A = \underbrace{\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} \overset{\lambda_1}{0} & 0 \\ 0 & \overset{\lambda_2}{-5} \end{pmatrix}}_{\Lambda} \underbrace{\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}^{-1}}_{V^{-1}}$

$\dfrac{d\vec{u}}{dt} = V\Lambda(V^{-1}\vec{u})$

or $\dfrac{d}{dt}\underbrace{(V^{-1}\vec{u})}_{\vec{y}} = \Lambda(V^{-1}\vec{u})$

$V^{-1}\vec{u} = \vec{y}$; Therefore $\vec{u} = \underbrace{\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}}_{V} \vec{y} = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e^{\overset{\lambda_1}{0}t} \\ e^{\overset{\lambda_2}{-5}t} \end{pmatrix}$

$= \begin{pmatrix} 3 - e^{-5t} \\ 2 + e^{-5t} \end{pmatrix}$ (assuming $\vec{u}(0) = \vec{0}$)

(Can verify that $\dfrac{d\vec{u}}{dt} = A\vec{u}$)

Higher Order (Ordinary) Differential Equations (ODEs)

Consider

$$a \frac{d^3y}{dt^3} + b\frac{d^2y}{dt^2} + c\frac{dy}{dt} + Ky = 0 \qquad \text{where } a, b, c, K \in \mathbb{R}$$

Introduce two "extra" equations

$$y' = \frac{dy}{dt} \quad \text{and} \quad y'' = \frac{dy'}{dt}$$

Suppose $a=1$

$$\begin{aligned} a\frac{dy''}{dt} &= -b\frac{dy'}{dt} - c\frac{dy}{dt} - Ky \\ \frac{dy'}{dt} &= y'' \\ \frac{dy}{dt} &= y' \end{aligned}$$

equivalent system

$$\frac{d}{dt}\begin{pmatrix} y'' \\ y' \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -b & -c & -K \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{} \begin{pmatrix} y'' \\ y' \\ y \end{pmatrix}$$

diagonalize this matrix and proceed as before

EXPONENTIAL OF A MATRIX

$\frac{d\vec{u}}{dt} = A\vec{u}$ has solution $\vec{u} = u(0)e^{At}$ <u>if</u> we can define what the exponential of a matrix is

<u>Note:</u> (Scalar) $e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \ldots$

(Matrix) $e^{At} := I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \ldots$

scalar

$A \in \mathbb{R}^{n \times n}$

$$= I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots$$

$$\frac{d}{dt}(e^{At}) = \frac{d}{dt}\left(I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 + \ldots\right)$$

$$= A + A^2 t + A^3\frac{t^2}{2} + \ldots$$

$$= A\Big[\underbrace{1 + At + A^2\frac{t^2}{2} + \ldots}_{e^{At}}\Big]$$

$$= A\,e^{At}$$

(works like the scalar case)

<u>Note:</u> Diagonalization of A will help us compute e^{At}