EIGENVALUES AND EIGENVECTORS

Def^n Suppose $A \in \mathbb{C}^{n\times n}$, then $\vec{x} \neq \vec{0}$, $\vec{x} \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$ (scalar) are eigenvectors and eigenvalues respectively if

$$A\vec{x} = \lambda \vec{x}$$

THEOREM (Eigenvalues of a matrix)

Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

(Proof sketch) Let λ be an eigenvalue of A

$\iff$ there exists $\vec{x} \neq 0$ so that $A\vec{x} = \lambda\vec{x}$

$\iff$ — " — $A\vec{x} - \lambda\vec{x} = \vec{0}$

$\iff$ — " — $A\vec{x} - \lambda I_n \vec{x} = \vec{0}$

$\iff$ — " — $(A - \lambda I_n)\vec{x} = \vec{0}$

$\iff$ $A - \lambda I_n$ is singular

$\iff$ $\det(A - \lambda I_n) = 0$

- the polynomial $p_A(\lambda) = \det(A - \lambda I_n)$ is the characteristic polynomial of A
- when A is $n \times n$, $p_A(\lambda)$ has degree n. hence A has n eigenvalues (note: repeats, complex values possible)

SUMMARY - THE EIGENVALUE PROBLEM

To solve the eigenvalue problem for an $n \times n$ matrix, follow these steps:

Step 1

Compute the determinant of $A - \lambda I$. (this is a polynomial of degree n in λ)

Step 2

Find the roots of this polynomial by solving $\det(A - \lambda I) = 0$.

The n roots are the n eigenvalues of A

Step 3

For each eigenvalue λ_i, solve $(A - \lambda_i I)\vec{x}_i = \vec{0}$ to find an eigenvector $\vec{x}_i$

(for $i = 1, 2, \ldots, n$)

EXAMPLE Find the eigenvalues and the corresponding eigenvectors of the matrix $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$

(Step 1) Compute $\det(A - \lambda I)$

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = (3-\lambda)(-2-\lambda) - 6$$
$$= -(3-\lambda)(2+\lambda) - 6$$
$$= -(6+\lambda-\lambda^2) - 6$$
$$= \lambda^2 - \lambda - 12$$

(Step 2) Determine eigenvalues using $|A - \lambda I| = 0$

We have $\lambda^2 - \lambda - 12 = 0$

$\Rightarrow (\lambda - 4)(\lambda + 3) = 0$

$\Rightarrow \lambda = 4$ or $\lambda = -3$

Therefore the eigenvalues are

$\lambda_1 = 4$ and $\lambda_2 = -3$

(Step 3) Find eigenvectors by solving $(A-\lambda I)\vec{x} = \vec{0}$

For $\lambda_1 = 4$, we get $A - \lambda I = A - 4I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Solving $(A-4I)\vec{x} = \vec{0}$, (performing elimination)

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \xrightarrow[R_2 \to R_2 + 3R_1]{} \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1/_{-1}} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

1 free col / variable (col 2)

Hence (special sol^n) $s_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

eigenvector $\vec{x}$ corr. to $\lambda = 4$ is $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Similarly, for $\lambda_2 = -3$, $A - \lambda I = A + 3I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

solving $(A+3I)\vec{x} = \vec{0}$, we get $\vec{x}_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

PROPERTIES (Let $A \in \mathbb{R}^{n \times n}$)

① If $\vec{x}$ is an eigenvector of A (corresponding to eigenvalue λ), then $\vec{x} = c\vec{x}$, $c \neq 0$ is also an eigenvector, with eigenvalue unchanged.

Why? Consider $A\vec{y} = A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x}) = \lambda\vec{y}$

② The eigenvalues of A^2 and A^{-1} are λ^2 and λ^{-1}, with the same eigenvectors.

Why? Consider $A^2\vec{x} = A(A\vec{x}) = A(\lambda\vec{x}) = \lambda(A\vec{x}) = \lambda^2\vec{x}$

Similarly $A^{-1}\vec{x} = A^{-1}\left(\frac{1}{\lambda}\lambda\vec{x}\right) = \frac{1}{\lambda}A^{-1}(\lambda\vec{x}) = \frac{1}{\lambda}A^{-1}(A\vec{x}) = \frac{1}{\lambda}I_n\vec{x} = \frac{1}{\lambda}\vec{x}$

Similar results hold for A^n and A^{-n}.

③ The product of the n eigenvalues equal to the determinant.

The sum of the eigenvalues equals the $\underbrace{\text{sum of the } n \text{ diagonal entries}}_{\text{trace}}$

EIGENVALUES / EIGENVECTORS OF SPECIAL MATRICES

SINGULAR MATRICES

(Example) Let $A = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$

Finding the eigenvalues of A, we have

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 2 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 8 = 8 - 6\lambda + \lambda^2 - 8 = \lambda^2 - 6\lambda$$

Setting $|A-\lambda I| = 0$, we get $\lambda(\lambda-6) = 0$ or $\lambda = 0$, $\lambda = 6$

<u>If A is singular, $\lambda = 0$ is an eigenvalue</u>. The eigenvectors fill $N(A)$.

Conversely if A is invertible zero is not an eigenvalue.

PROJECTION MATRICES

The projection matrix $P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0$ with corresponding eigenvectors $\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

Note:

- the eigenvalues for $\lambda = 0$ fill the nullspace while the eigenvalues for $\lambda = 1$ fill the columnspace.
- P is singular; so $\lambda = 0$ is an eigenvalue.
- P is symmetric, so its eigenvectors are perpendicular (we will see this later on too)
- P is a Markov matrix – this guarantees $\lambda = 1$ is an eigenvalue

  entries are positive and columns add to 1

Triangular Matrices

the eigenvalues of a triangular matrix lie on its diagonal.

(Example) $A = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ then $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 \\ 2 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda)$

Setting $|A - \lambda I| = 0$ gives $\underline{\lambda_1 = 1}$ and $\underline{\lambda_2 = 3}$

Rotation Matrices

Consider $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (a 90° rotation)

- Q has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$
- the corresponding eigenvectors are $\vec{x}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} i \\ 1 \end{pmatrix}$

Note: $i = \sqrt{-1}$

Note:

- Q is real, but $\lambda, \vec{x}$ are imaginary
- no vector $Q\vec{x}$ stays in the same direction as $\vec{x}$ (Q is a rotation matrix); hence no real eigenvectors.
- Q is orthogonal; this requires $|\lambda| = 1$ for each λ.
- Q is skew symmetric; such matrices have purely imaginary λ.

$Q^T = -Q$