LECTURE 27

11/7/16

LAST TIME

$\det(\cdot) : M_{n\times n} \longrightarrow \mathbb{R}$ with properties

① $\mathrm{Det}(I) = 1$

② The determinant changes sign when two rows are interchanged

③ The determinant is a linear function of each row separately;

i.e. $\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$

Note: all rules apply to columns also (since $|A| = |A^T|$)

TODAY

* Permutations and cofactors

Different ways of computing Determinants

Pivot Formula

(Recall) Every matrix $A \in \mathbb{R}^{n \times n}$ has an LU decomposition

$$PA = LU$$

upper triangular matrix

permutation matrix made up of q row swaps

lower triangular matrix with 1's on the diagonal

Pivot Formula $|A| = (-1)^q |U|$

Since the determinant of a triangular matrix is the product of its diagonal elements

$$\det A = \pm 1 \, (d_1 \, d_2 \cdots\cdots d_n)$$ where d_i denotes pivot values

EXAMPLE:

Find det A if $A = \begin{pmatrix} 2 & 3 & -7 \\ 0 & 0 & 2 \\ 0 & -1 & 6 \end{pmatrix}$

Exchanging rows 2 and 3, we have

$$PA = \begin{pmatrix} 2 & 3 & -7 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \quad \text{where} \quad P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$|PA| = \det \underbrace{\begin{pmatrix} 2 & 3 & -7 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix}}_{\text{upper triangular matrix}}$$

$$\Rightarrow \quad |P|\,|A| = (2)(-1)(2) = -4$$

$$\Rightarrow \quad |A| = 4 \quad \text{since} \quad |P| = -1.$$

Note:

$$\begin{bmatrix} A_1\ \boxed{a_{11}} & a_{12} & a_{13} & a_{14} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots & & \\ a_{n1} & \cdots & \cdots & \cdots & & a_{nn} \end{bmatrix}$$

A_2 A_3

If there are no row exchanges,

- the first pivot $d_1 = a_{11}$ (depends only on the left corner matrix A_1)
- the second pivot d_2 depends only on A_2

$\vdots$

- the k^{th} pivot d_k depends only on A_k

$$\det A_k = d_1 d_2 \cdots d_k$$

$\Rightarrow$ Pivots from determinants: $d_k = \dfrac{|A_k|}{|A_{k-1}|} = \dfrac{d_k d_{k-1} \cdots d_1}{d_{k-1} d_{k-2} \cdots d_1}$

THE "BIG" FORMULA FOR DETERMINANTS

- the pivot formula is fast to compute for large n, but it is also difficult for theoretical analysis since they are not easily given in terms of the original a_{ij}'s

Recall ($n=2$) $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ (2 terms (2! terms))

- In general, the big formula has $n!$ terms

Where does this formula come from?

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + (-1)\begin{vmatrix} c & d \\ 0 & b \end{vmatrix} = ad - bc$$

$$= \begin{vmatrix} a & 0 \\ c+0 & d+0 \end{vmatrix} + (-1)\begin{vmatrix} c+0 & 0+d \\ 0 & b \end{vmatrix}$$

$$= \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$$

$$+ (-1)\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix} + (-1)\begin{vmatrix} 0 & d \\ 0 & b \end{vmatrix}$$

$$= ac\underbrace{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}_{0} + ad\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + (-1)\,bc\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$+ (-1)bd\underbrace{\begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}}_{0}$$

$$= ad - bc$$

When $n=3$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{23} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

3x3 Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{array}{l} + a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{array}$$

In general, $\det A =$ sum over all $n!$ possible (column) permutations $P(\alpha, \beta, \dots, \omega)$

$$= \sum (\det P)\, a_{1\alpha} a_{2\beta} \cdots a_{n\omega}$$

$\boxed{\text{DETERMINANT BY COFACTORS}}$

(Recall)

3x3 Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{aligned} &+ a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ &- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{aligned}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactors

$\boxed{\text{Defn}}$ Let $A \in \mathbb{R}^{n \times n}$. Then Cofactor $c_{ij} = (-1)^{i+j} \det M_{ij}$

where $M_{ij} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a matrix obtained by deleting row i and column j from A

$$\boxed{\det A = \sum_{j=1}^{n} a_{ij}\, c_{ij} \quad \text{for any } A \in \mathbb{R}^{n\times n} \text{ for any row } i}$$

Moreover, since $\det A = \det A^T$

$$\boxed{\det A = \sum_{i=1}^{n} a_{ij}\, c_{ij} \quad \text{for any column } j}$$

Note: We can use the cofactor formula by expanding along any row or any column.

(computationally convenient to choose the one with most 0's)

EXAMPLE: Find $|A|$ when $A = \begin{bmatrix} 3 & 5 & 0 & 1 \\ -2 & 1 & 2 & 0 \\ 0 & 4 & 0 & 0 \\ 7 & 0 & 5 & 2 \end{bmatrix}$

we can use $|A| = 3\begin{vmatrix} 1 & 2 & 0 \\ 4 & 0 & 0 \\ 0 & 5 & 2 \end{vmatrix} - 5\begin{vmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 7 & 5 & 2 \end{vmatrix} + 0\cdot\left|\quad\right| - 1\begin{vmatrix} -2 & 1 & 2 \\ 0 & 4 & 0 \\ 7 & 0 & 5 \end{vmatrix}$

but I'm lazy!

Instead use $|A| = 4\,M_{32}$ (cofactor expansion along row 3)

$$= 4\,(-1)^{3+2}\cdot\begin{vmatrix} 3 & 0 & 1 \\ -2 & 2 & 0 \\ 7 & 5 & 2 \end{vmatrix}$$

$$= -4\left(3\begin{vmatrix} 2 & 0 \\ 5 & 2 \end{vmatrix} + 1\begin{vmatrix} -2 & 2 \\ 7 & 5 \end{vmatrix}\right) = -4\left(3(4) + (-10-14)\right)$$

$$= -4\,(12 - 24) = 48$$