DETERMINANTS

What is the determinant of a matrix?

The determinant of an $n \times n$ matrix, $A \in \mathbb{R}^{n \times n}$ is a real number denoted by $\det A$ or $|A|$.

Determinants are complicated functions of matrices. They are difficult to write down!

We will instead calculate determinants using their properties

PROPERTIES OF DETERMINANTS

Here, we list the 4 most important properties

① If $U = \begin{pmatrix} u_{11} & u_{12} & \cdots & \cdots \\ & u_{22} & \cdots & \cdots \\ & & \ddots & \\ 0 & & & u_{nn} \end{pmatrix}$ is upper triangular, then

$$|U| = u_{11}\, u_{22} \cdots u_{nn}.$$

The determinant of an upper triangular matrix is the product of its diagonal elements

② If $L \in \mathbb{R}^{n\times n}$ is lower triangular, then

$$|L| = l_{11}\, l_{12} \cdots l_{nn}$$

$$L = \begin{pmatrix} l_{11} & & & 0 \\ l_{21} & l_{22} & & \\ \cdots & \cdots & \ddots & \\ \cdots & \cdots & \cdots & l_{nn} \end{pmatrix}$$

The determinant of a lower triangular matrix is the product of its diagonal elements.

PROPERTIES OF DETERMINANTS

③ Let $A, B \in \mathbb{R}^{n\times n}$. Then $|AB| = |A|\,|B|$.

④ Let $S_{i\leftrightarrow j}$ be the swap or row exchange matrix that exchanges rows i and j. Then

$$|S_{i\leftrightarrow j}| = \begin{cases} -1 & \text{if } i \neq j \\ 1 & \text{else} \end{cases}$$

Example: $S_{1\leftrightarrow 2} \in \mathbb{R}^{3\times 3}$ is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (identity matrix with rows 1 and 2 swapped)

$|S_{1\leftrightarrow 2}| = -1$

$\rightsquigarrow$ these rules separately imply that $|I| = 1$

Identity matrix

COMPUTING THE DETERMINANT

We know that every $A \in \mathbb{R}^{n \times n}$ has an LU decomposition after reordering its rows

$$PA = LU$$

P: permutation matrix with $P = S_{i_1 \leftrightarrow j_1} S_{i_2 \leftrightarrow j_2} \cdots S_{i_q \leftrightarrow j_q}$

L: lower triangular with 1's on its diagonal

U: upper triangular

Using the 4 rules above,

$$|PA| = |LU| \xrightarrow{\text{rule 3}} |P||A| = |L||U|$$

$$\Rightarrow (-1)^q |A| = 1\,|U| \qquad \left(\begin{array}{l}\text{rule 4 - determinant of } S_{i \leftrightarrow j} \\ \text{rule 2 - det of } L\end{array}\right)$$

$$\Rightarrow |A| = (-1)^q |U|$$

U is the upper triangular form obtained on elimination

Determinant of a 2x2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Performing elimination, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & b \\ 0 & d - \left(\frac{c}{a}\right) b \end{bmatrix}$

Upper triangular

Using rule ①, $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a\left(d - \left(\frac{c}{a}\right) b\right)$

$= ad - bc$

EXAMPLE: Compute the determinant of $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

Strategy: Use elimination to reduce A to upper triangular form (counting row swaps along the way)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[\substack{R_3 - 3R_1 \\ R_4 - 4R_1}]{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & -2 & -8 & -10 \\ 0 & -7 & -10 & -13 \end{bmatrix} \xrightarrow[\substack{R_3 - 2R_2 \\ R_4 - 7R_2}]{} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 4 & 36 \end{bmatrix} \xrightarrow[R_4 + R_3]{} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -7 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 40 \end{bmatrix}}_{U}$$

- there were no row exchanges required

$$|A| = (-1)^0 |U|$$

$$= 1(-1)(-4)(40)$$

$$|A| = 160$$

DETERMINANTS AND INVERTIBILITY

Suppose $A \in \mathbb{R}^{n\times n}$ has rank $r < n$. What is $|A|$?

$$A \xrightarrow{\text{elimination}} \underbrace{\begin{pmatrix} \ddots & \cdots \\ 0 & \ddots \end{pmatrix}}_{\substack{\text{upper triangular matrix} \\ \text{with at least one zero} \\ \text{on the diagonal}}} \Longrightarrow \det A = |A| = 0!$$

$$\boxed{|A| \neq 0 \text{ if and only if } A \text{ is full rank if only if } A \text{ is invertible}}$$

Suppose $A \in \mathbb{R}^{n\times n}$ is invertible. Then

$$A^{-1}A = I \iff |A^{-1}A| = |I| = 1$$

$$\iff |A^{-1}|\,|A| = 1 \quad \text{(by rule 3)}$$

Hence $\boxed{|A^{-1}| = \dfrac{1}{|A|}}$

$$\boxed{\text{DETERMINANT OF } A^T}$$

Let $A \in \mathbb{R}^{n \times n}$.

We have $A = \underbrace{\left(S_{i_1 \leftrightarrow j_1} S_{i_2 \leftrightarrow j_2} \cdots S_{i_q \leftrightarrow j_q} \right)}_{\substack{\text{Swap matrices} \\ \text{(recall: these are symmetric)}}} L\, U$ (LU factorization)

Therefore $A^T = U^T L^T \left(S_{i_1 \leftrightarrow j_1} S_{i_2 \leftrightarrow j_2} \cdots S_{i_q \leftrightarrow j_q} \right)$ (using $(ABC)^T = C^T B^T A^T$ and $(S_{i \leftrightarrow j})^T = S_{i \leftrightarrow j}$)

$$\Rightarrow \quad |A^T| = |U^T| \cdot 1 \cdot (-1)^q$$ (using rule 3 and structure of L (1's on diagonal))

$$= |U| (-1)^q$$

$$\boxed{|A^T| = |A|}$$

OTHER IMPORTANT PROPERTIES

* Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. Consider rescaling a row by $c \in \mathbb{R}$

Then $\left| \begin{pmatrix} ca_{11} & ca_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| = \left| \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \right| = \left| \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} A \right|$

$= c|A|$

In general, rescaling by a scalar 'c' rescales the determinant by c.

* adding one row to another

Let $c \in \mathbb{R}$

$$\left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21}+ca_{11} & a_{22}+c\,a_{12} \end{pmatrix} \right| = \left| \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} A \right| = \left| \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \right| |A| = |A|$$