FINDING ORTHONORMAL BASES - GRAM-SCHMIDT

Suppose we want an orthonormal basis for span$\{\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n\}$. We can build one using Gram-Schmidt procedure.

Procedure

Step 0 $\quad \vec{q}_1 = \dfrac{\vec{a}_1}{\|\vec{a}_1\|}$ (normalizing to unit length)

Step 1 $\quad \vec{q}_2 = \dfrac{\vec{a}_2 - P_{\text{span}\{\vec{q}_1\}}\vec{a}_2}{\left\| \vec{a}_2 - P_{\text{span}\{\vec{q}_1\}}\vec{a}_2 \right\|}$ we have $P_{\text{span}\{\vec{q}_1\}}\vec{a}_2 = \vec{p}_1 = (\vec{q}_1 \cdot \vec{a}_2)\,\vec{q}_1$

Step 2 $\quad \vec{q}_3 = \dfrac{\vec{a}_3 - P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3}{\left\| \vec{a}_3 - P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3 \right\|}$ where $P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3 = \vec{p}_2 = (\vec{q}_1 \cdot \vec{a}_3)\,\vec{q}_1 + (\vec{q}_2 \cdot \vec{a}_3)\,\vec{q}_2$

...

continue

EXAMPLE: Find an orthonormal basis for the column space of $A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

Step 0

$$r_{11} = \| \vec{a}_1 \| = \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\| = 2.$$

$$\vec{q}_1 = \frac{\vec{a}_1}{r_{11}} = \frac{\vec{a}_1}{\| \vec{a}_1 \|} \quad \text{or} \quad \boxed{\vec{q}_1 = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}.}$$

Recall:

$$\vec{q}_{k+1} = \frac{\vec{a}_{k+1} - P_{\text{span}\{\vec{q}_1, \dots, \vec{q}_k\}} \vec{a}_{k+1}}{\left\| \vec{a}_{k+1} - P_{\text{span}\{\vec{q}_1, \dots, \vec{q}_k\}} \vec{a}_{k+1} \right\|}$$

where $P_{\text{span}\{\vec{q}_1, \dots, \vec{q}_k\}} = \vec{P}_k$

$$= (\vec{a}_{k+1} \cdot \vec{q}_1)\vec{q}_1 + \dots + \underbrace{(\vec{a}_{k+1} \cdot \vec{q}_k)}_{r_{k,k+1}}\vec{q}_k$$

Step 1

$$r_{12} = (\vec{a}_2 \cdot \vec{q}_1) = \vec{q}_1^{\,T} \vec{a}_2 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} = 3$$

Now, $\vec{p}_1 = r_{12} \vec{q}_1 = 3\vec{q}_1$

we have
$$\vec{a}_2 - \vec{p}_1 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} 3/2 \\ 3/2 \\ 3/2 \\ 3/2 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5/2 \\ 5/2 \\ -5/2 \end{pmatrix}$$

$$r_{22} = \| \vec{a}_2 - \vec{p}_1 \| = 5$$

$$\vec{q}_2 = \frac{\vec{a}_2 - \vec{p}_1}{\| \vec{a}_2 - \vec{p}_1 \|} = \frac{1}{5}\left(-\tfrac{5}{2}, \tfrac{5}{2}, \tfrac{5}{2}, -\tfrac{5}{2}\right) \quad \text{or} \quad \boxed{\vec{q}_2 = \begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{pmatrix}}$$

Step 3

$$r_{13} = (\vec{a}_3 \cdot \vec{q}_1) = \vec{q}_1^{\,T}\vec{a}_3 = \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)\begin{pmatrix}4\\-2\\2\\0\end{pmatrix} = 2$$

$$r_{23} = (\vec{a}_3 \cdot \vec{q}_2) = \vec{q}_2^{\,T}\vec{a}_3 = -2$$

and $\vec{p}_2 = P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3 = r_{13}\vec{q}_1 + r_{23}\vec{q}_2 = 2\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix} - 2\begin{pmatrix}-1/2\\1/2\\1/2\\-1/2\end{pmatrix} = \begin{pmatrix}2\\0\\0\\2\end{pmatrix}$

we now have $\vec{a}_3 - \vec{p}_2 = \begin{pmatrix}4\\-2\\2\\0\end{pmatrix} - \begin{pmatrix}2\\0\\0\\2\end{pmatrix} = \begin{pmatrix}2\\-2\\2\\-2\end{pmatrix}$

$$r_{33} = \|\vec{a}_3 - \vec{p}_2\| = 4$$

$$\vec{q}_3 = \frac{\vec{a}_3 - \vec{p}_2}{\|\vec{a}_3 - \vec{p}_2\|} = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} \quad \text{or} \quad \boxed{\vec{q}_3 = \begin{pmatrix}1/2\\-1/2\\1/2\\-1/2\end{pmatrix}}$$

$$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$$

- Q is an orthogonal matrix
- Its columns are an orthonormal basis (ONB) for $C(A)$

QR FACTORIZATION

Any $m \times n$ matrix with independent columns can be factored into QR.

- the matrix Q has orthonormal columns
- the square matrix R which upper triangular with positive entries on the diagonal.

In the previous problem, we have

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}}_{R}$$

Why do we care about QR?

Consider least squares solutions

$$(A^T A \hat{\vec{x}} = A^T \vec{b}) \xrightarrow[A = QR]{\text{use}} (QR)^T (QR) \hat{\vec{x}} = (QR)^T \vec{b}$$

$$\Rightarrow R^T \underbrace{(Q^T Q)}_{I} R \hat{\vec{x}} = R^T Q^T \vec{b} \Rightarrow R \hat{\vec{x}} = Q^T \vec{b}$$

upper triangular

or

$$\hat{\vec{x}} = R^{-1} Q^T \vec{b}$$

solve by back substitution

② This 4×4 Hadamard matrix is an orthogonal matrix.

$$Q = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix}$$

(a) What projection matrix P_4 will project every $\vec{b}$ in $\mathbb{R}^4$ onto the line through $\vec{q}_4$?

(b) What projection matrix P_{123} will project every $\vec{b}$ in $\mathbb{R}^4$ onto the subspace spanned by $\vec{q}_1, \vec{q}_2$ and $\vec{q}_3$?

(c) Suppose A is the 4×3 matrix whose columns are $\vec{q}_1, \vec{q}_2, \vec{q}_3$.
Find the least squares solution $\hat{x}$ to

$$Ax = b \iff \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

(a) Let $A = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

Now consider the projection matrix onto the $C(A)$ given by

$$P = A(A^TA)^{-1}A^T$$

Note that $A^TA = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 4$

and $(A^TA)^{-1} = \frac{1}{4}$

Hence $P = A(A^TA)^{-1}A^T = \frac{1}{4} AA^T = \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$

$$= \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$