LECTURE 23

10/28/2016

## LAST TIME

* Least-Squares
* Fitting data points to a polynomial

## TODAY

* Orthonormal basis
* Gram-Schmidt procedure
* QR factorization

# ORTHONORMAL BASIS

**Defn** An orthonormal basis $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$ is a set of vectors with

$$\vec{q}_i \cdot \vec{q}_j = \vec{q}_i^{\,T} \vec{q}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Each vector, $\vec{q}_j$ in the orthonormal basis (ONB) must have length 1 since $\vec{q}_i \cdot \vec{q}_j = \|\vec{q}_j\| = 1$.

Example $\left\{ \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\}$ is an orthonormal basis for $\mathbb{R}^2$.

# ORTHOGONAL MATRIX

**Def$^n$**

An orthogonal matrix, $Q \in \mathbb{R}^{n\times n}$, is a matrix whose column vectors are an orthonormal basis.

If $Q = \begin{pmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{pmatrix}$,

then $Q^TQ = \begin{pmatrix} — \vec{q}_1 — \\ — \vec{q}_2 — \\ \vdots \\ — \vec{q}_n — \end{pmatrix}\begin{pmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_n \\ | & | & & | \end{pmatrix}$

$$= \begin{pmatrix} \underbrace{\vec{q}_1\cdot\vec{q}_1}_{1} & \cancelto{0}{\vec{q}_1\cdot\vec{q}_2} & \cdots & \cdots \\ \cancelto{0}{\vec{q}_2\cdot\vec{q}_1} & \underbrace{\vec{q}_2\cdot\vec{q}_2}_{1} & \vec{q}_2\cdot\vec{q}_3 & \cdots \\ & & \ddots & \\ \vec{q}_n\cdot\vec{q}_1 & \vec{q}_n\cdot\vec{q}_1 & \cdots & \vec{q}_n\cdot\vec{q}_n \end{pmatrix} = I \in \mathbb{R}^{n\times n}$$

# Consequences of Orthogonality

① If $Q \in \mathbb{R}^{n \times n}$ is orthogonal, then $Q^T = Q^{-1}$ !

(Why? $Q^T Q = I$
$\Rightarrow (Q^T Q) Q^{-1} = I Q^{-1}$
$\Rightarrow Q^T (Q Q^{-1}) = Q^{-1}$
$\Rightarrow Q^T = Q^{-1}$)

② If $Q \in \mathbb{R}^{n \times n}$ is orthogonal, then

$$\begin{aligned}
\|Q\vec{x}\|^2 = (Q\vec{x}) \cdot (Q\vec{x}) &= (Q\vec{x})^T (Q\vec{x}) \\
&= \vec{x}^T \underbrace{(Q^T Q)}_{I} \vec{x} \\
&= \vec{x}^T \vec{x} \\
&= \vec{x} \cdot \vec{x} \\
&= \|\vec{x}\|^2
\end{aligned}$$

Hence, $Q$ preserves the length of all vectors $\vec{x} \in \mathbb{R}^n$

# CONSEQUENCES OF ORTHOGONALITY

③ Let $Q \in \mathbb{R}^{n\times n}$ be orthogonal. Then $Q\vec{x} = \vec{b}$ has solution $\vec{x} = Q^T\vec{b}$

Why? $Q\vec{x} = \vec{b}$

$$\Rightarrow Q^{-1}(Q\vec{x}) = Q^{-1}\vec{b}$$

$$\Rightarrow \underbrace{(Q^{-1}Q)}_{I}\vec{x} = Q^{-1}\vec{b}$$

$$\Rightarrow \vec{x} = Q^{-1}\vec{b} = Q^T\vec{b}$$

$$\vec{x} = \begin{pmatrix} \vec{q}_1 \cdot \vec{b} \\ \vdots \\ \vec{q}_n \cdot \vec{b} \end{pmatrix}$$

Hence $\vec{x} = \sum_{j=1}^{n} (\vec{q}_j \cdot \vec{x})\, \vec{q}_j \quad \forall \vec{x} \in \mathbb{R}^n$

$\rightsquigarrow$ Rewriting $\vec{x}$ as a sum of an orthonormal basis

④ Let $Q \in \mathbb{R}^{m\times n}$ be orthogonal; then projecting onto $C(Q)$ is simple.

(Projection matrix) $P = Q\underbrace{(Q^TQ)}_{I}^{-1}Q^T = QQ^T$

EXAMPLE  Project $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto $\text{span}\left\{\begin{pmatrix}2\\0\\2\end{pmatrix}\right\}$

$$\text{Span}\left\{\begin{pmatrix}2\\0\\2\end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}\right\}$$

this vector has length 1

it is an ONB for the 1D subspace

The projection of $\begin{pmatrix}1\\1\\1\end{pmatrix}$ onto $\text{span}\left\{\begin{pmatrix}2\\0\\2\end{pmatrix}\right\}$ is $P\begin{pmatrix}1\\1\\1\end{pmatrix}$ where $P = QQ^T$ and $Q = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

Hence

$$\underbrace{\underbrace{\begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}}_{Q}\underbrace{\begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\end{pmatrix}}_{Q^T}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}\sqrt{2} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

Projection matrix is $P = \begin{pmatrix}1/2 & 0 & 1/2\\0 & 0 & 0\\1/2 & 0 & 1/2\end{pmatrix}$

## FINDING ORTHONORMAL BASES - GRAM-SCHMIDT

Suppose we want an orthonormal basis for $\text{span}\{\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n\}$. We can build one using Gram-Schmidt procedure.

### PROCEDURE

**Step 0**

$$\vec{q}_1 = \frac{\vec{a}_1}{\|\vec{a}_1\|}$$

(normalizing to unit length)

**Step 1**

$$\vec{q}_2 = \frac{\vec{a}_2 - P_{\text{span}\{\vec{q}_1\}}\vec{a}_2}{\left\| \vec{a}_2 - P_{\text{span}\{\vec{q}_1\}}\vec{a}_2 \right\|}$$

we have $P_{\text{span}\{\vec{q}_1\}}\vec{a}_2 = (\vec{q}_1 \cdot \vec{a}_2)\,\vec{q}_1$

**Step 2**

$$\vec{q}_3 = \frac{\vec{a}_3 - P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3}{\left\| \vec{a}_3 - P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3 \right\|}$$

where $P_{\text{span}\{\vec{q}_1, \vec{q}_2\}}\vec{a}_3 = (\vec{q}_1 \cdot \vec{a}_3)\,\vec{q}_1 + (\vec{q}_2 \cdot \vec{a}_3)\,\vec{q}_2$

...

continue