LECTURE 21

10/25/2016

LAST TIME

* Orthogonal subspaces
* Orthogonal complements
* Orthogonality of the four subspaces

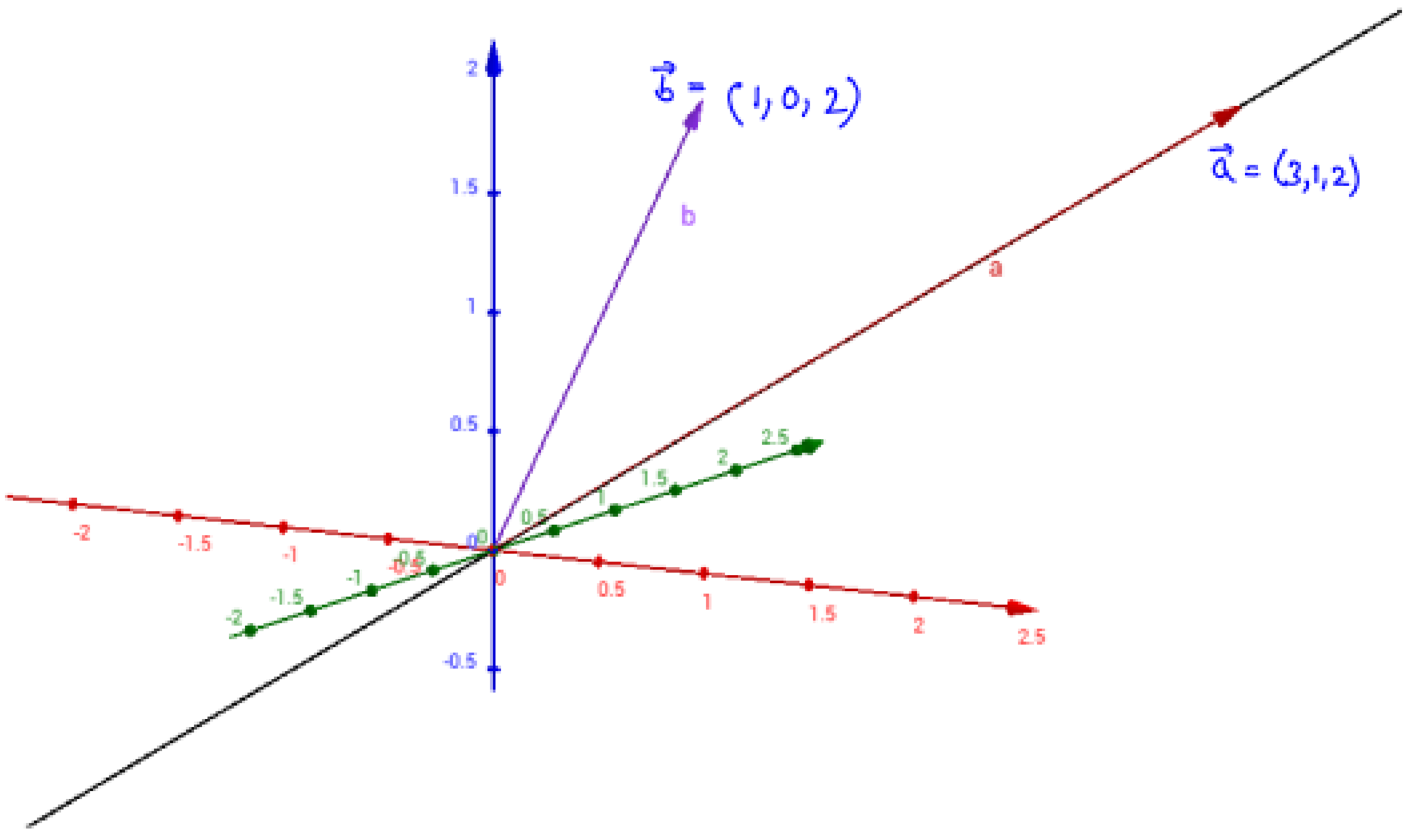
TODAY

* Projections
* Normal equations
* Projection matrices

Projection Onto A Line

An Example.....

Find the projection of $\vec{b} = (1,0,2)$ onto $\vec{a} = (3,1,2)$



FINDING THE PROJECTION OF $\vec{b}$ ONTO $\vec{a}$

INGREDIENTS

- line going through $\vec{a} = (a_1, a_2, \dots, a_m)$
- given point $\vec{b} = (b_1, b_2, \dots, b_m)$

Note: projection $\vec{p}$ is the part of $\vec{b}$ along the line through $\vec{a}$

KEY IDEA

The line from $\vec{b}$ to $\vec{p}$ is perpendicular to the vector $\vec{a}$

$$\Rightarrow \quad \vec{a} \cdot \vec{e} = 0$$

$$\Rightarrow \quad \vec{a} \cdot (\vec{b} - \hat{x}\,\vec{a}) = 0$$

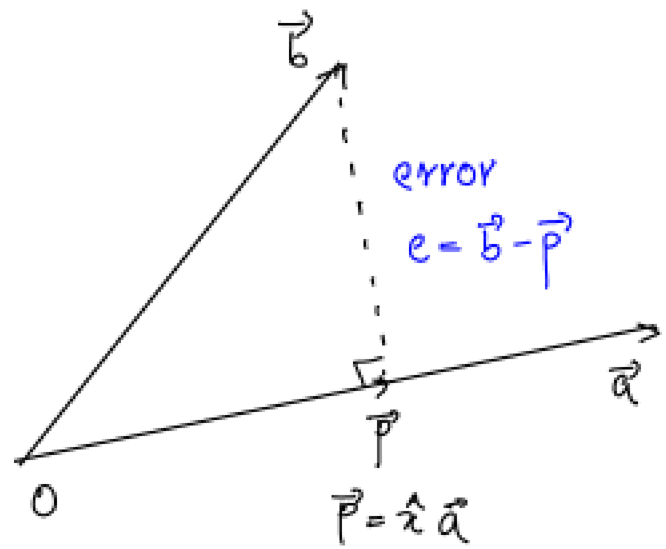
$$\Rightarrow \quad \vec{a} \cdot \vec{b} - \hat{x}\,\vec{a} \cdot \vec{a} = 0$$

$$\Rightarrow \quad \hat{x} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} = \frac{\vec{a}^T \vec{b}}{\vec{a}^T \vec{a}}$$

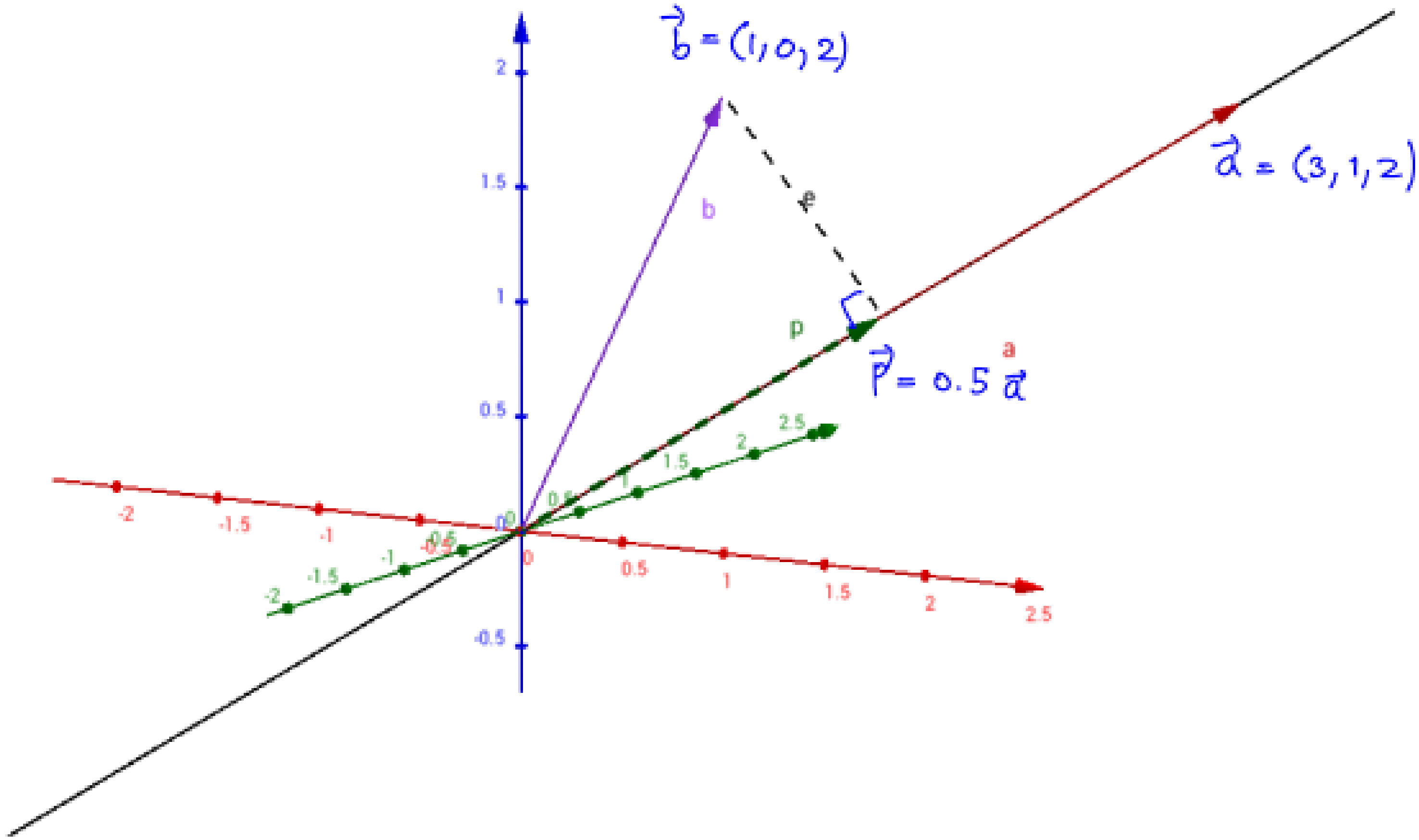
NOTATION

* denote the projection by $\vec{p}$
* we'll denote $\vec{p} = \hat{x}\,\vec{a}$
* projection matrix – P

  we have $\vec{p} = P\vec{b}$



FINDING THE PROJECTION OF $\vec{b}$ ONTO $\vec{a}$



PROJECTION OF $\vec{b}$ ONTO THE LINE THROUGH $\vec{a}$

$$\boxed{\hat{x} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}}$$

Projection of $\vec{b}$ onto the line through $\vec{a}$ is the vector $\boxed{\vec{p} = \hat{x}\,\vec{a} = \left(\frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}}\right)\vec{a}}$

Projection Matrix $\boxed{P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}}}$

Note:
- P is rank-1 with $P^2 = P$
- we are projecting onto a one-dimensional subspace (line)
- this line is the column space of P

Special cases
1. $\vec{b} = \vec{a}$ $\quad(\hat{x} = 1,\ \vec{p} = \vec{a})$
2. $\vec{b}$ is perpendicular to $\vec{a}$ $\quad(\hat{x} = 0,\ \vec{p} = \vec{0})$

PROJECTION ONTO A SUBSPACE

PROBLEM STATEMENT Let $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n \in \mathbb{R}^m$ be linearly independent. Find the combination $\vec{p} = \hat{x}_1 \vec{a}_1 + \hat{x}_2 \vec{a}_2 + \ldots\ldots + \hat{x}_n \vec{a}_n$ closest to a given vector $\vec{b}$.

* Construct matrix A whose columns are $\vec{a}_1, \vec{a}_2, \ldots\ldots, \vec{a}_n$

* We are looking for the particular combination of the columns $\vec{p} = A\hat{\vec{x}}$ that is closest to $\vec{b}$.

KEY IDEA

- Our subspace is the column space of A
- Like the projection onto a line, we want the error vector $\vec{b} - A\hat{\vec{x}}$ to be perpendicular to all vectors in the column space

$\Rightarrow$ $\vec{b} - A\hat{\vec{x}}$ is in the left nullspace, or $A^T(\vec{b} - A\hat{\vec{x}}) = \vec{0}$

PROJECTION ONTO A SUBSPACE

Note: $A \in \mathbb{R}^{m \times n}$

$\vec{p} = \hat{x}_1 \vec{a}_1 + \dots + \hat{x}_n \vec{a}_n = A\vec{x}$

To find $\hat{x}$ (vector, $n \times 1$)

Solve $\boxed{A^T(\vec{b} - A\hat{\vec{x}}) = 0 \quad \text{or} \quad \underline{A^T A \hat{\vec{x}} = A^T \vec{b}}}$

normal equations

$A^T A \in \mathbb{R}^{n \times n}$, invertible if $\vec{a}_i$ are independent

$$\underline{\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b}}$$

Projection of $\vec{b}$ onto the subspace is $\vec{p}$ (vector, $m \times 1$)

$$\boxed{\vec{p} = A\hat{\vec{x}} = A(A^T A)^{-1} A^T \vec{b}}$$

Projection matrix, $P \in \mathbb{R}^{m \times m}$

$$\boxed{P = A(A^T A)^{-1} A^T}$$

Example Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Find $\hat{x}$, $\vec{p}$ and P.

- Start by computing A^TA

$$A^TA = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$$

- Now compute $A^T\vec{b}$ $\quad A^T\vec{b} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

- Solve normal equation $(A^TA)\hat{x} = A^T\vec{b}$

$$\begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

elimination gives $\left[\begin{array}{cc|c} 6 & 3 & 2 \\ 3 & 5 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 6 & 3 & 2 \\ 0 & 7/2 & 0 \end{array}\right] \Rightarrow \hat{x}_2 = 0, \ \hat{x}_1 = 1/3$

$$\underline{\hat{x} = (0, 1/3)}$$

- $\vec{p} = A\hat{x} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix}$

Note: $\vec{e} = \vec{b} - \vec{p} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -4/3 \\ 1/3 \\ 2/3 \end{pmatrix}$ $\quad \vec{e}$ is perpendicular to both columns of A

Example Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix}$ and $\vec{b} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$. Find $\hat{x}$, $\vec{p}$ and P.

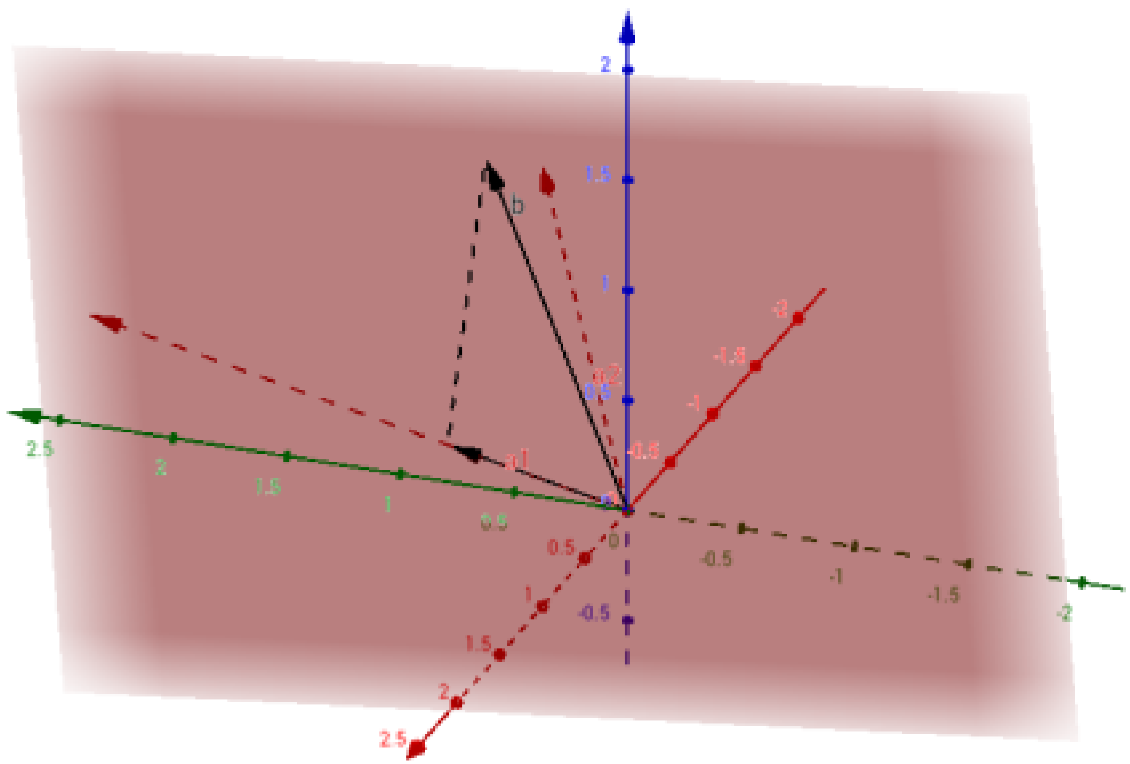
$$A^T A = \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \qquad (A^T A)^{-1} = \frac{1}{21} \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$$

$$P = A (A^T A)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \left(\frac{1}{21} \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \right) \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 10 & -1 \\ 3 & -6 & 9 \end{bmatrix}$$

$$= \frac{1}{21} \begin{bmatrix} 5 & 4 & 8 \\ 4 & 20 & -2 \\ 8 & -2 & 17 \end{bmatrix}$$

Projection Onto A Subspace



$\boxed{\text{PROPERTIES OF } P}$

$\boxed{\text{Theorem}}$ A^TA is invertible if $A \in \mathbb{R}^{m \times n}$ has rank n

or

A^TA is invertible if A has linearly independent columns.

$$* \quad P^2 = PP = \left(A(A^TA)^{-1}A^T\right)\left(A(A^TA)^{-1}A^T\right)$$

$$= A\,(A^TA)^{-1}\underbrace{(A^TA)\,(A^TA)^{-1}}_{=I}A^T)$$

$$= A\,(A^TA)^{-1}A^T$$

$$= P$$

$$* \quad P^T = \left(A\,(A^TA)^{-1}A^T\right)^T = A\left((A^TA)^{-1}\right)^TA^T = A\left((A^TA)^T\right)^{-1}A^T$$

$$= A\,(A^TA)^{-1}A^T$$

$$= P$$

$\Rightarrow P$ is symmetric