LECTURE 20

10/21/2016

Until now . . .

- Solve $A\vec{x} = \vec{b}$
  - Chapter 2: find some solution using LU decomposition or finding A^{-1}
  - Chapter 3: find all possible solutions

In both cases, we assumed $\vec{b} \in C(A)$

Now . . . . . (Chapter 4)

- Solve $A\vec{x} = \vec{b}$ approximately when $\vec{b} \notin C(A)$

Numerous application in engineering, statistics . . . .

GEOMETRY OF THE FUNDAMENTAL SUBSPACES

An example .....

Find the four fundamental subspaces of $A = \begin{bmatrix} 1 & 5 & -1 \\ 2 & 10 & 3 \\ \pi & 5\pi & 0 \end{bmatrix}$.

Performing elimination, we get

$$[A \,\vdots\, \vec{b}\,] = \left[\begin{array}{ccc:c} 1 & 5 & -1 & b_1 \\ 2 & 10 & 3 & b_2 \\ \pi & 5\pi & 0 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - \pi R_1]{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc:c} 1 & 5 & -1 & b_1 \\ 0 & 0 & 5 & b_2 - 2b_1 \\ 0 & 0 & \pi & b_3 - \pi b_1 \end{array}\right] \xrightarrow[R_3 \to R_3/\pi]{R_2 \to R_2/5} \left[\begin{array}{ccc:c} 1 & 5 & -1 & b_1 \\ 0 & 0 & 1 & \frac{b_2}{5} - \frac{2}{5}b_1 \\ 0 & 0 & 1 & \frac{b_3}{\pi} - b_1 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{ccc:c} 1 & 5 & -1 & b_1 \\ 0 & 0 & 1 & \frac{b_2}{5} - \frac{2}{5}b_1 \\ 0 & 0 & 0 & \frac{b_3}{\pi} - \frac{3}{5}b_1 - \frac{1}{5}b_2 \end{array}\right] \xrightarrow{R_1 \to R_1 + R_2} \left[\begin{array}{ccc:c} 1 & 5 & 0 & \frac{3}{5}b_1 + \frac{b_2}{5} \\ 0 & 0 & 1 & \frac{b_2}{5} - \frac{2}{5}b_1 \\ 0 & 0 & 0 & \frac{b_3}{\pi} - \frac{3}{5}b_1 - \frac{b_2}{5} \end{array}\right]$$

pivot cols

Geometry of the Fundamental Subspaces

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pivot rows

pivot cols

coefficients say what combination give the zero row

$$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ \pi \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

$$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$N(A) = \text{span}\left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Since free var $= x_2$

$$N(A^T) = \text{span}\left\{ \begin{pmatrix} -3/5 \\ -1/5 \\ 1/\pi \end{pmatrix} \right\}$$

Note: can also do any scalar multiple

ex: $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ -5/\pi \end{pmatrix} \right\}$

GEOMETRY OF THE FUNDAMENTAL SUBSPACES

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OBSERVATIONS

basis vector(s) for $N(A)$ and $C(A^T)$ are perpendicular to each other

$(-5, 1, 0) \cdot (1, 5, 0) = 0$

$(-5, 1, 0) \cdot (0, 0, 1) = 0$

Consider generic vectors

$\vec{n} \in N(A)$

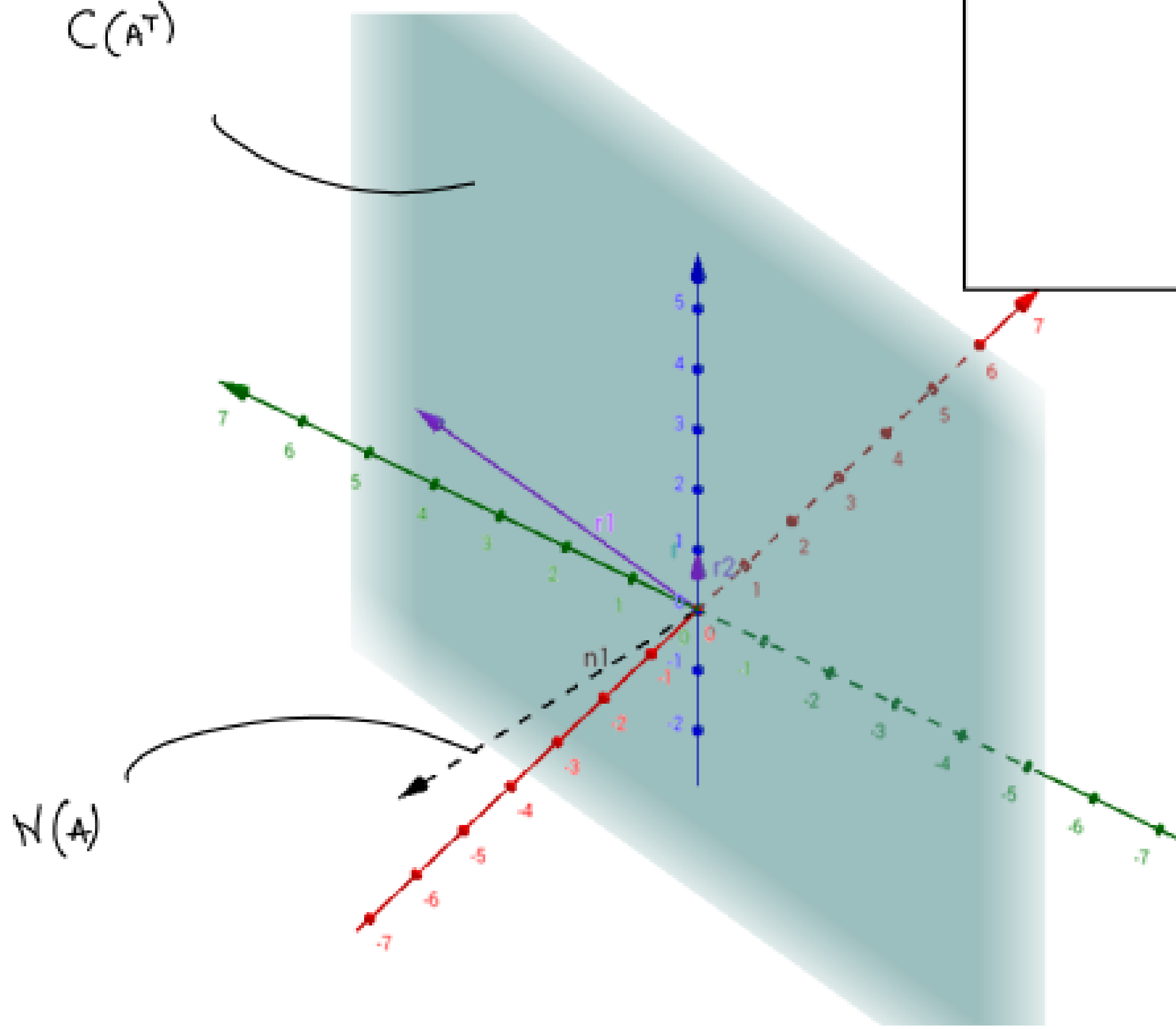
$\vec{r} \in C(A^T)$

$\vec{n} = \lambda_0 \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} \qquad \lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$

$\vec{r} = \lambda_1 \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\vec{n} \cdot \vec{r} = (-5\lambda_0, \lambda_0, 0) \cdot (\lambda_1, 5\lambda_1, \lambda_2)$

$= -5\lambda_0\lambda_1 + 5\lambda_0\lambda_1 = 0$



$$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

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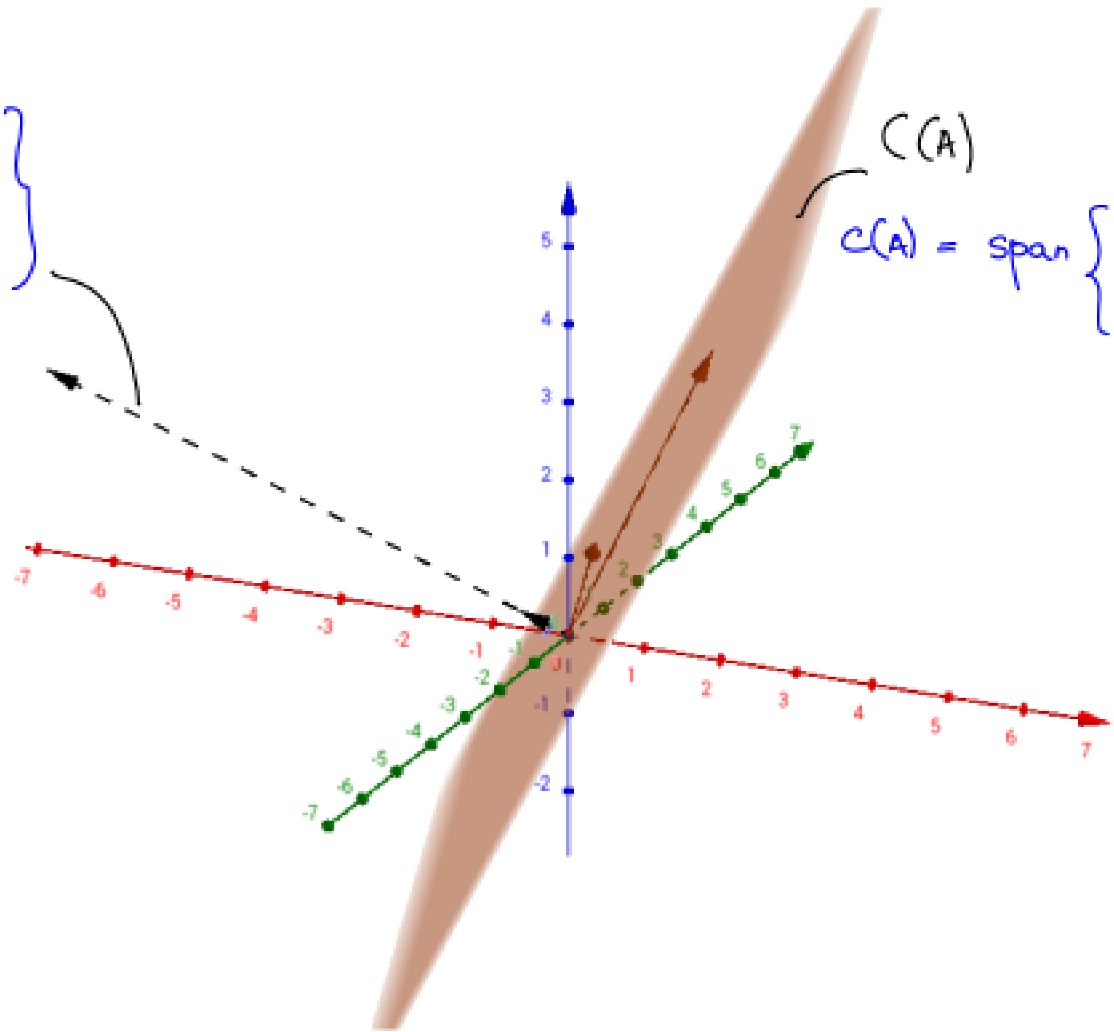

$$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ \pi \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

GEOMETRY OF THE FUNDAMENTAL SUBSPACES

Similarly basis vector(s) for $N(A^T)$ and $C(A)$ are perpendicular to each other

It follows that any vector in $N(A^T)$ is perpendicular to any vector in $C(A)$.

$$\boxed{C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ \pi \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \right\} \quad N(A^T) = \text{span} \left\{ \begin{pmatrix} -3/5 \\ -1/5 \\ 1/\pi \end{pmatrix} \right\}}$$

Defn (ORTHOGONAL SUBSPACES)

Two subspaces V and W of a vector space are <u>orthogonal</u> if every vector $\vec{v} \in V$ is perpendicular to every vector $\vec{w} \in W$.

i.e., $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w} = 0$ for all $\vec{v} \in V$, $\vec{w} \in W$.

<u>Notation:</u>

$W \perp V$

We have

Nullspace $N(A)$ and Row space $C(A^T)$ are orthogonal subspaces of $\mathbb{R}^n$
Left Nullspace $N(A^T)$ and Column Space $C(A)$ are orthogonal subspaces of $\mathbb{R}^m$

$\boxed{\text{Thm}}$ Subspaces W and V will be orthogonal if they both have a basis

$$W = \text{span}\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_r\} \text{ and } V = \text{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_s\} \text{ with}$$

$$\vec{w}_i \cdot \vec{v}_j = 0 \quad \text{for all } i = 1, 2, \ldots, r \text{ and } j = 1, 2, \ldots, s.$$

<u>Proof:</u>

ORTHOGONAL COMPLEMENTS

Defⁿ The orthogonal complement of subspace V contains every vector that is perpendicular to V.

This orthogonal subspace is denoted by $\underbrace{V^{\perp}}$

pronounced "V perp"

FUNDAMENTAL THEOREM OF LINEAR ALGEBRA (PART 2)

The nullspace $N(A)$ is the orthogonal complement of the row space $C(A^T)$ (in $\mathbb{R}^n$)

The left nullspace $N(A^T)$ is the orthogonal compliment of the column space $C(A)$ (in $\mathbb{R}^m$)

Why is this useful?

Every $\vec{x}$ can be split into $\underbrace{\vec{x}_r}_{\text{row space component}} + \underbrace{\vec{x}_n}_{\text{nullspace component}}$

$\vec{x} = \vec{x}_r + \vec{x}_n$

Recall: A basis for a vector space satisfies two properties

- the basis vectors are linearly independent,
- the basis vectors span the space.

As a consequence,

* Any n independent vectors in $\mathbb{R}^n$ must span $\mathbb{R}^n$. So they are a basis
* Any n vectors that span $\mathbb{R}^n$ must be independent. So they are a basis.

Let $A \in \mathbb{R}^{m\times n}$

We have $\underbrace{N(A)}_{\dim\, n-r} \perp \underbrace{C(A^T)}_{\dim\, r}$ $\qquad \mathbb{R}^n = \text{span}\{N(A) \cup C(A^T)\}$

Similarly,

$\underbrace{N(A^T)}_{\dim\, m-r} \perp \underbrace{C(A)}_{\dim\, r}$ $\qquad \mathbb{R}^m = \text{span}\{N(A^T) \cup C(A)\}$