Problem Set - Chapter 3

① Bases for the fundamental subspaces.

Find a basis for each fundamental subspace of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}$

Let $\vec{b} = (b_1, b_2, b_3)$. Perform elimination on $[A \,\vdots\, \vec{b}]$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 2 & b_1 \\ 1 & 2 & 1 & 3 & b_2 \\ 3 & 6 & 3 & 7 & b_3 \end{array}\right] \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left[\begin{array}{cccc:c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 1 & b_3 - 3b_1 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{cccc:c} 1 & 2 & 1 & 2 & b_1 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \left[\begin{array}{cccc:c} 1 & 2 & 1 & 0 & 3b_1 - 2b_2 \\ 0 & 0 & 0 & 1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right]$$

Pivot columns: 1 and 4

Free columns: 2 and 3

Pivot rows: 1 and 2

Special sol^{n}s.

$\vec{s}_1 = (-2, 1, 0, 0)$

$\vec{s}_2 = (-1, 0, 1, 0)$

free vars are x_2, x_3

Basis for $C(A)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix} \right\}$

Basis for $C(A^T)$ is $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis for $N(A)$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

Basis for $N(A^T)$ is $\left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$

② Complete Solution of $A\vec{x}=\vec{b}$

Find all possible solutions to $A\vec{x}=\vec{b}$ when $A=\begin{bmatrix}1&0&1&0\\0&2&2&1\\4&-2&2&-1\end{bmatrix}$ and $\vec{b}=\begin{bmatrix}2\\-10\\18\end{bmatrix}$

Performing elimination on $[A \,\vdots\, \vec{b}]$, we have

$$\left[\begin{array}{cccc|c}1&0&1&0&2\\0&2&2&1&-10\\4&-2&2&-1&18\end{array}\right] \xrightarrow[R_3\to R_3-4R_1]{} \left[\begin{array}{cccc|c}1&0&1&0&2\\0&2&2&1&-10\\0&-2&-2&-1&10\end{array}\right] \xrightarrow[R_3\to R_3+R_2]{} \left[\begin{array}{cccc|c}1&0&1&0&2\\0&2&2&1&-10\\0&0&0&0&0\end{array}\right]$$

$$\xrightarrow[\substack{R_2\to R_2/2\\ R_3\to R_3/4}]{} \left[\begin{array}{cccc|c}1&0&1&0&2\\0&1&1&\frac{1}{2}&-5\\0&0&0&0&0\end{array}\right]$$

pivot cols are 1 and 2
free cols (vars) are 3 and 4

Particular sol^n set $x_3=x_4=0$. then $\vec{x}_p=\begin{pmatrix}2\\-5\\0\\0\end{pmatrix}$

(Note) $x_1+x_3=2$
$x_2+x_3+\frac{1}{2}x_4=-5$

Nulspace sol^n there are 2 free vars.

Set $x_3=1$, $x_4=0$; we get $\vec{s}_1=(-1,-1,1,0)$
set $x_3=0$, $x_4=1$; we get $\vec{s}_2=(0,-\frac{1}{2},0,1)$

$\vec{x}_n = x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}0\\-\frac{1}{2}\\0\\1\end{pmatrix}$ where $x_3,x_4\in\mathbb{R}$.

Hence, the complete/general solution to $A\vec{x}=\vec{b}$ is $\vec{x}=\vec{x}_p+\vec{x}_n = \begin{pmatrix}2\\-5\\0\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}+x_4\begin{pmatrix}0\\-\frac{1}{2}\\0\\1\end{pmatrix}$

③ Matrix with specified column space and null space

Construct a matrix whose column space contains $(1,1,5)$ and $(0,3,1)$ and whose nullspace contains $(1,1,2)$

Step 1: Determine size of matrix: If $A \in \mathbb{R}^{m \times n}$, we know $C(A) \subseteq \mathbb{R}^m$ and $N(A) \subseteq \mathbb{R}^n$

This tell us that $m = n = 3$

Step 2: Construct matrix $A = \begin{bmatrix} 1 & 0 & c_1 \\ 1 & 3 & c_2 \\ 5 & 1 & c_3 \end{bmatrix}$ Why? since $\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \in C(A)$

Step 3: Perform elimination

$$\begin{bmatrix} 1 & 0 & c_1 \\ 1 & 3 & c_2 \\ 5 & 1 & c_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c_1 \\ 0 & 3 & c_2 - c_1 \\ 0 & 1 & c_3 - 5c_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & \frac{c_2 - c_1}{3} \\ 0 & 1 & c_3 - 5c_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c_1 \\ 0 & 1 & (c_2 - c_1)/3 \\ 0 & 0 & d \end{bmatrix}$$

$$\begin{aligned} d &= (c_3 - 5c_1) \\ &\quad - (c_2 - c_1)/3 \\ &= c_3 - \frac{c_2}{3} - \frac{14}{3} c_1 \end{aligned}$$

we have reached reduced row echelon form

Step 4: Special solution $\vec{s}_1 = \begin{bmatrix} -c_1 \\ -(c_2 - c_1)/3 \\ 1 \end{bmatrix}$ Let $\alpha \cdot \vec{s}_1 = (1, 1, 2)$ we get $\boxed{\begin{matrix} c_1 = -\frac{1}{2} \\ c_2 = -2 \end{matrix}}$

and we need $d = 0$

$$\alpha \begin{bmatrix} -c_1 \\ (c_1 - c_2)/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

then set $d = 0$ to get $\boxed{c_3 = -3}$

④ Construct a matrix A which has $\begin{bmatrix}1\\1\\2\end{bmatrix}$ as a basis for its column space and $\begin{bmatrix}0\\-1\\3\\2\end{bmatrix}$ as a basis for its row space.

- this says that all columns are linear combinations of $\begin{pmatrix}1\\1\\2\end{pmatrix}$; all rows are linear combinations of $\begin{pmatrix}0\\-1\\3\\2\end{pmatrix}$
- What type of matrix has this property? - Rank-1 matrix

$$A = \underbrace{\begin{bmatrix}1\\1\\2\end{bmatrix}}_{u} \underbrace{\begin{bmatrix}0 & -1 & 3 & 2\end{bmatrix}}_{v^T} = \begin{bmatrix}0 & -1 & 3 & 2\\0 & -1 & 3 & 2\\0 & -2 & 6 & 4\end{bmatrix}$$

- any scalar multiple of this matrix will also work.

⑤ Basis for $\mathbb{R}^n$

Are the vectors $\begin{pmatrix}0\\1\\1\end{pmatrix}$, $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\3\\2\end{pmatrix}$ a basis for the vector space $\mathbb{R}^3$?

Let $A = \begin{bmatrix}0 & 1 & 1\\1 & 1 & 3\\1 & 1 & 2\end{bmatrix}$. Performing elimination,

$\boxed{\text{Yes}}$

$$\begin{bmatrix}0 & 1 & 1\\1 & 1 & 3\\1 & 1 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1 & 3\\0 & 1 & 1\\1 & 1 & 2\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 1 & 3\\0 & 1 & 1\\0 & 0 & -1\end{bmatrix}$$

We have a full set of pivots. Hence all columns are pivot columns.

Since $r=3$, columns are independent, and $C(A)$ is all of $\mathbb{R}^3$

Some Short Answer Questions

① Suppose $\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is the only solution to $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$,

the columns of A span a $\underline{\;3\;}$ dimensional subspace of $\mathbb{R}^{\underline{5}}$

② Suppose a 3 by 4 matrix A has rank 3. Then the equation $A\vec{x} = \vec{b}$

(always / sometimes) has (unique / many / no) solutions.

③ Let $A = \begin{bmatrix} 0 & 0 \\ 6 & 9 \\ 2 & 3 \end{bmatrix}$

- The column space A is (point / line / plane / 3D space)
- The row space of A is a vector subspace of $\mathbb{R}^3$ True / False
- Circle best answer
  - A has full column rank
  - A has full row rank
  - Neither of the above