LECTURE 18

10/17/2016

LAST TIME

* Linear independence
* Basis
* Dimension

TODAY

* Dimensions of the Four Subspaces

FOUR FUNDAMENTAL SUBSPACES

Let $A \in \mathbb{R}^{m \times n}$

1. The ROW SPACE is $C(A^T)$. It is a subspace of $\mathbb{R}^n$

2. The COLUMN SPACE is $C(A)$. — " — of $\mathbb{R}^m$

3. The NULLSPACE is $N(A)$ — " — of $\mathbb{R}^n$

4. The LEFT NULLSPACE is $N(A^T)$ — " — of $\mathbb{R}^m$

<u>Note:</u> The ROWSPACE AND COLUMNSPACE have dimension r = rank of the matrix

The NULLSPACE, $N(A)$, has dimension $n-r$.

The LEFT NULLSPACE has dimension $m-r$.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{\text{ELIMINATION}} R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(reduad row echelon form)

Here $m = 3$

$n = 4$

$r = 2$

pivot rows – 1 and 2

pivot cols – 1 and 3

We have:

Row Space

Dimension is the rank r

Nonzero rows of R form basis

$C(A^T) = C(R^T)$

Dimension $= r = 2$

rows 1 and 2

Column Space

Dimension is the rank r

Pivot cols (of A) form a basis

$C(A) \neq C(R)$

Dimension $= r = 2$

cols 1 and 3

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{\text{ELIMINATION}} R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(reduced row echelon form)

Here $m = 3$

$n = 4$

$r = 2$

pivot rows – 1 and 2

pivot cols – 1 and 3

We have: NULLSPACE

Dimension is $n-r$

Special solutions form a basis

$N(A) = N(R)$

Dimension $= n - r = 2$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{s}_2 = \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

$\vec{x}_n = x_2 \vec{s}_1 + x_4 \vec{s}_2, \quad x_2 x_4 \in \mathbb{R}$

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{\text{ELIMINATION}} R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(reduad row echelon form)

Here $m = 3$

$n = 4$

$r = 2$

pivot rows – 1 and 2

pivot cols – 1 and 3

We have : LEFT NULLSPACE Dimension is $m - r$ | Dimension $= m - r = 1$

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix} \xrightarrow{\text{ELIMINATION}} R = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(reduced row echelon form)

Here $m=3$

$n=4$

$r=2$

pivot rows - 1 and 2

pivot cols - 1 and 3

LEFT NULLSPACE Dimension is $m-r$

Dimension $= m-r = 1$

We have:

$N(A^T)$ We want solutions to $A^T\vec{x} = \vec{0}$

Perform elimination on $[A \vdots \vec{b}]$ to give $[R \vdots \vec{d}]$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & b_1 \\ 2 & 4 & -3 & 0 & b_2 \\ 1 & 2 & 1 & 5 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & b_1 \\ 0 & 0 & -1 & -2 & b_2-2b_1 \\ 0 & 0 & 2 & 4 & b_3-b_1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & b_1 \\ 0 & 0 & -1 & -2 & b_2-2b_1 \\ 0 & 0 & 0 & 0 & b_3-b_1+2(b_2-2b_1) \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & b_1 \\ 0 & 0 & 1 & 2 & 2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_3+2b_2-5b_1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 3b_1-b_2 \\ 0 & 0 & 1 & 2 & 2b_1-b_2 \\ \hline 0 & 0 & 0 & 0 & b_3+2b_2-5b_1 \end{array}\right]$$

basis for $N(A^T)$ is $\left\{ \begin{pmatrix} -5 \\ 2 \\ 1 \end{pmatrix} \right\}$

$\boxed{\text{FUNDAMENTAL THEOREM OF LINEAR ALGEBRA}}$

Let A be an $m \times n$ matrix

(PART I) $\boxed{\begin{array}{l}\text{The column space and row space both have dimension } r. \\ \text{The nullspaces have dimensions } n-r \text{ and } m-r.\end{array}}$

Alternatively,

$$\dim C(A) + \dim N(A) = n$$

(Sometimes called rank-nullity theorem)

$$\dim C(A^T) + \dim N(A^T) = m$$

<u>EXAMPLE:</u> Let $A = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 3 \\ 3 & 6 & 3 & 7 \end{bmatrix}$. Find a basis for each of the fundamental subspaces and find their respective dimension