LECTURE 16

10 | 12 | 2016

LAST TIME

* Review - Nullspace, $N(A)$
* Rank of a matrix
* Rank-1 matrices

TODAY

* Complete solution to $A\vec{x} = \vec{b}$
* Special cases: full column rank
  full row rank

All Solutions to $A\vec{x} = \vec{b}$

$$\underbrace{\begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}}_{A} \vec{x} = \vec{b}$$

First, note that $A\vec{x} = \vec{b}$ has a solution only if $\vec{b} \in C(A)$

Consider $\vec{b} = \begin{pmatrix} -6 \\ -4 \end{pmatrix}$.

Performing elimination on $[A \vdots \vec{b}]$, we have

$$\left[\begin{array}{cc:c} 3 & 3 & -6 \\ 2 & 2 & -4 \end{array}\right] \xrightarrow[-\frac{2}{3} R_1 + R_2]{} \left[\begin{array}{cc:c} 3 & 3 & -6 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1/3} \left[\begin{array}{cc:c} \boxed{1} & 1 & \boxed{\begin{matrix} -2 \\ 0 \end{matrix}} \\ 0 & 0 & \end{array}\right]$$

pivot

free

$x_1 + x_2 = -2$

(1 free variable) Special solution $\vec{s}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (solution to $A\vec{x} = \vec{0}$)

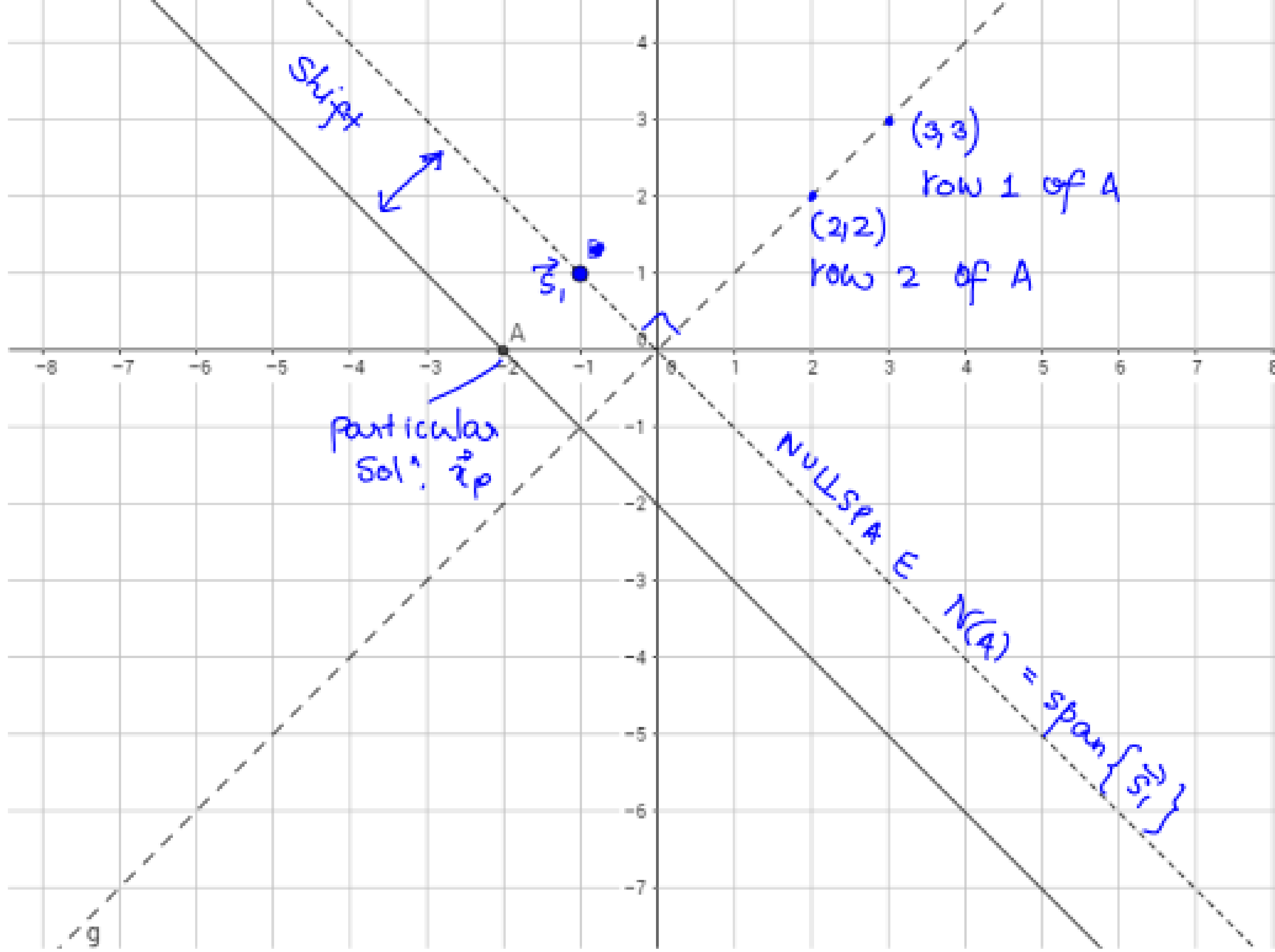
$$\left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$\boxed{\text{Particular}}$ Solution Choose $\underline{x_2 = 0}$, then $x_1 = -2$; i.e., $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

Complete solution is $\boxed{\vec{x} = \vec{x}_p + \vec{x}_n = \begin{pmatrix} -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Notes



- All linear combinations of rows of A are perpendicular to N(A)
- the particular solution tells you where to shift/slide N(A)
- all solutions to $A\vec{x}=\vec{b}$ form a "hyperplane" (the nullspace N(A), n−r dimensional) shifted so that it passes through a point given by the particular sol^n.

$\vec{x}_{nullspace}$	n−r special solutions to $A\vec{x}_n=\vec{0}$
$\vec{x}_{particular}$	Particular solution solves $A\vec{x}_p=\vec{b}$

Complete solution $\begin{pmatrix} \text{one } \vec{x}_p \\ \text{many } \vec{x}_n \end{pmatrix}$ $\vec{x}=\vec{x}_p+\vec{x}_n$

Example Find all solutions to $\begin{bmatrix} 1 & 2 & 7 \\ -1 & 0 & -1 \\ 3 & -1 & 0 \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

Perform elimination on the augmented matrix $[A \,\vdots\, \vec{b}]$

$$\left[\begin{array}{ccc:c} 1 & 2 & 7 & 3 \\ -1 & 0 & -1 & -1 \\ 3 & -1 & 0 & 2 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 + R_1} \left[\begin{array}{ccc:c} 1 & 2 & 7 & 3 \\ 0 & 2 & 6 & 2 \\ 0 & -7 & -21 & -7 \end{array}\right] \xrightarrow[R_3/-7]{R_2/2} \left[\begin{array}{ccc:c} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{array}\right] \xrightarrow[R_3 - R_1]{} \left[\begin{array}{ccc:c} 1 & 2 & 7 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc:c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivot cols   free col

Nullspace $N(A)$

(1 special sol^n) $\vec{s}_1 = \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$. Hence $N(A) = x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$ for all $x_3 \in \mathbb{R}$

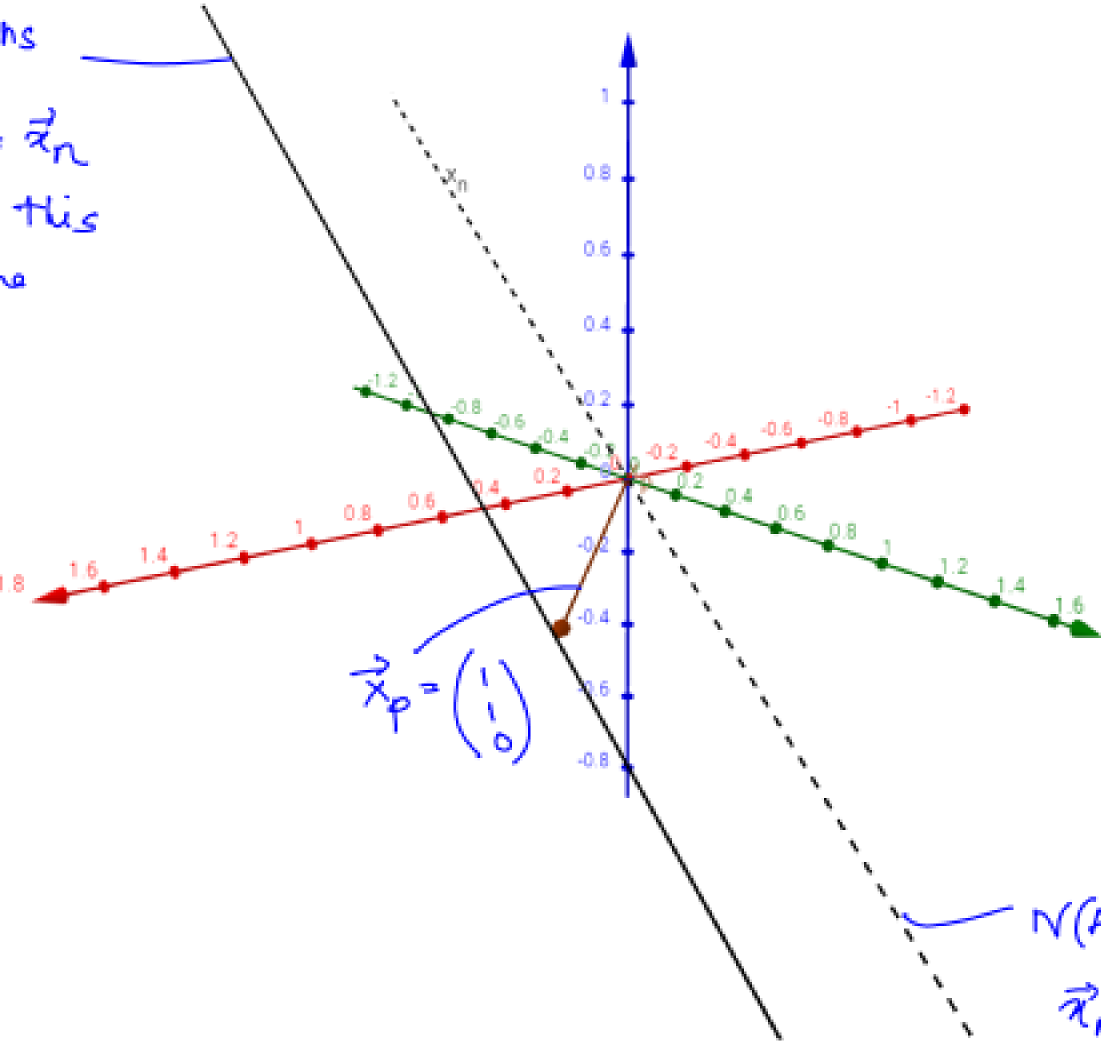
Particular solution choose $x_3 = 0$; $\vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

Complete solution $\boxed{\vec{x} = \vec{x}_p + \vec{x}_n = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}}$

All solutions

$\vec{x} = \vec{x}_p + \vec{x}_n$

lie on this line



$$\vec{x}_p = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$N(A)$

$$\vec{x}_n = x_3 \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}$$

PROCEDURE FOR FINDING ALL SOLUTIONS TO $A\vec{x} = \vec{b}$

① Form augmented matrix $[A \,\vdots\, \vec{b}]$. Use elimination to find reduced row echelon form.

② When does the system have a solution(s)?

In reduced row echelon form,

$$\left(\begin{array}{c|c} \tilde{R} & \tilde{b}_1 \\ & \vdots \\ \hline \boxed{\vec{0}\text{s}} & \boxed{\tilde{b}_m} \end{array}\right) = [R \,\vdots\, d]$$

No soln if the zeros do not correspond

③ Particular solution is $\begin{pmatrix} \tilde{b}_1 \\ \vdots \\ \tilde{b}_m \end{pmatrix}$ with 0's put in at the free variable spots.

④ $N(A)$ given by linear combinations of special solutions; special solutions obtained from free columns of R.

$\boxed{\text{SPECIAL CASES}}$

$\boxed{\mathrm{I}}$ $A\vec{x}=\vec{b}$, $A \in \mathbb{R}^{n\times n}$, rank r (square matrix)

① $r=n$ A is invertible
- $A\vec{x}=\vec{b}$ has 1 solution, $\vec{x}=A^{-1}\vec{b}$
- $N(A)=\{\vec{0}\}$

② $r<n$ (not full rank)

$\vec{b} \notin C(A) \Rightarrow$ no solution

$\vec{b} \in C(A) \Rightarrow$ inf. solutions $\vec{x}_p + N(A)$

- $N(A)$ has $n-r$ "free" special solutions that span it

$\boxed{\text{SPECIAL CASES}}$

$\boxed{\text{II}}$

$A \in \mathbb{R}^{m\times n}$, $m > n$

$$\underbrace{A}_{m \times n} \; \underbrace{\vec{x}}_{n} = \underbrace{\vec{b}}_{m}$$

- A is tall and thin
- more equations than variables (overdetermined system)

① $r = n$ A is full column rank

- All columns of A are pivot columns.
- No free variables | special solutions
- $N(A) = \{\vec{0}\}$
- $\vec{b} \notin C(A) \Rightarrow$ no solution

  $\vec{b} \in C(A) \Rightarrow$ only one solution

② $r < n$ not full rank

- $\vec{b} \notin C(A) \Rightarrow$ no solution
- $\vec{b} \in C(A) \Rightarrow$ infinitely many sol's.

SPECIAL CASES

$\boxed{\text{III}}$ $A \in \mathbb{R}^{m\times n}$, $m<n$

$$m\left\{\overbrace{\boxed{\qquad A \qquad}}^{n}\right. \boxed{\vec{x}}\Big\}n = \boxed{\vec{b}}\Big\}m$$

- A is short and wide
- fewer equations than unknowns (underdetermined system)

① $r=m$ A is full row rank

- All rows have pivots; R has no zero rows
- $A\vec{x}=\vec{b}$ has solution for every rhs. $\vec{b}$
- $C(A) = \mathbb{R}^m$
- $N(A)$ is spanned by $n-m$ special solutions (infinitely many solutions)

② $r<m$ not full rank

- $\vec{b} \notin C(A) \Rightarrow$ no solution
- $\vec{b} \in C(A) \Rightarrow$ infinitely many sol'ns.