LECTURE 15

10/10/2016

LAST TIME

* Nullspace of a matrix
* Special solutions
* Reduced row-echelon form

TODAY

* Rank of a matrix

REVIEW — NULL SPACE

Recall: $N(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$

$N(A)$ is a subspace.

How do we compute $N(A)$

* Using elimination, convert A to reduced echelon form

Note: Reduced echelon form of A is

- upper triangular
- pivot columns have a single 1, the rest of the entries are all zero.

Example: Let $A = \begin{bmatrix} 2 & 1 & 4 & 4 \\ 3 & 2 & 6 & 7 \\ 1 & 3 & 2 & 7 \end{bmatrix}$

Performing elimination, we have

$$A = \begin{bmatrix} 2 & 1 & 4 & 4 \\ 3 & 2 & 6 & 7 \\ 1 & 3 & 2 & 7 \end{bmatrix} \xrightarrow[\substack{\text{row}\,2 \,\to\, -\frac{3}{2}\,\text{row}\,1 + \text{row}\,2 \\ \text{row}\,3 \,\to\, \text{row}\,3 - \frac{1}{2}\,\text{row}\,1}]{} \begin{bmatrix} 2 & 1 & 4 & 4 \\ 0 & \frac{1}{2} & 0 & 1 \\ 0 & \frac{5}{2} & 0 & 5 \end{bmatrix}$$

$$\xrightarrow[\text{row}\,3 \,\to\, -5\,\text{row}\,2 + \text{row}\,3]{} \begin{bmatrix} 2 & 1 & 4 & 4 \\ 0 & \frac{1}{2} & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\text{values}]{\text{divide by pivot}} \begin{bmatrix} 1 & \frac{1}{2} & 2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{row}\,1 \,\to\, -\frac{1}{2}\,\text{row}\,2 + \text{row}\,1} \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot cols

"free" cols.

$$A = \begin{bmatrix} 2 & 1 & 4 & 4 \\ 3 & 2 & 6 & 7 \\ 1 & 3 & 2 & 7 \end{bmatrix} \qquad R = \begin{bmatrix} \boxed{\begin{matrix}1\\0\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\\0\end{matrix}} & \begin{matrix}2\\0\\0\end{matrix} & \begin{matrix}1\\2\\0\end{matrix} \end{bmatrix}$$

$A \in \mathbb{R}^{3 \times 4}$

pivot cols "free" cols.

Given matrix — Reduced Echelon form

Notes

- there are 2 pivots / pivot columns
- (column 3) = 2 (colum 1)

  (column 4) = 1 (colum 1) + 2 (colum 2)
- "free" columns are a linear combination of earlier pivot columns.

Def^n (RANK) The rank of A (denoted by r) is the number of pivots

Easy to see that $r \leq m$ and $r \leq n$. In our example, rank$(A) = r = 2$.

$$A = \begin{bmatrix} 2 & 1 & 4 & 4 \\ 3 & 2 & 6 & 7 \\ 1 & 3 & 2 & 7 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$A \in \mathbb{R}^{3 \times 4}$

pivot cols "free" cols.

Given matrix

Reduced Echelon form

Column Space of A

$$C(A) = \text{span}\left\{ \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \subseteq \mathbb{R}^3$$

Nullspace of A

Special solutions $\quad \vec{s}_1 = \begin{pmatrix} -2 \\ 0 \\ \boxed{1} \\ 0 \end{pmatrix} \quad \vec{s}_2 = \begin{pmatrix} -1 \\ -2 \\ 0 \\ \boxed{1} \end{pmatrix}$.

Pivot vars: x_1, x_2

Free vars: x_3, x_4

Note: $R\vec{x} = \vec{0} \Rightarrow$

$$x_1 + 2x_3 + x_4 = 0$$
$$x_2 + 2x_4 = 0$$

$x_3 = 1,\ x_4 = 0 \rightarrow x_2 = 0,\ x_1 = -2$

$x_3 = 0,\ x_4 = 1 \rightarrow x_2 = -2,\ x_1 = -1$

Check that $R\vec{s}_1 = A\vec{s}_1 = \vec{0}$,

$R\vec{s}_2 = A\vec{s}_2 = \vec{0}$.

Hence $N(A) = \text{span}\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^4$.

Remarks

- All columns of $A \in \mathbb{R}^{m \times n}$ are either free or pivot columns

$$\underbrace{\#\text{ columns}}_{n} = \underbrace{\#\text{ pivot columns}}_{\substack{r \\ (\text{Rank of } A)}} + \underbrace{\#\text{ free columns}}_{n-r}$$

- $N(A)$ is always "spanned" by $n-r$ vectors (corresponding to the number of free columns).

$$A = \begin{bmatrix} \boxed{\begin{matrix} 2 \\ 3 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}} & \begin{matrix} 4 \\ 6 \\ 2 \end{matrix} & \begin{matrix} 4 \\ 7 \\ 7 \end{matrix} \end{bmatrix}$$

$A \in \mathbb{R}^{3 \times 4}$

Given matrix

$$R = \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \\ 0 \quad 0 & 0 \quad 0 \end{bmatrix}$$

pivot cols "free" cols.

Reduced Echelon form

(when the first r cols are the pivot cols)

$$R = \begin{bmatrix} I_{2\times 2} & F \\ 0 & 0 \end{bmatrix} \begin{matrix} r \text{ pivot rows} \\ m-r \text{ zero rows} \end{matrix}$$

r pivot cols | n−r free cols.

Nullspace matrix

$$N = \begin{bmatrix} \vec{s}_1 & \vec{s}_2 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -F \\ I \end{bmatrix} \begin{matrix} r \text{ pivot vars.} \\ n-r \text{ free vars} \end{matrix}$$

(n−r independent special sol^ns)

Example Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -1 & -3 & 2 \end{bmatrix}$

* find reduced row echelon form
* find $C(A)$ and $N(A)$
* determine rank of A

Elimination gives

$$\underset{A}{\begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -1 & -3 & 2 \end{bmatrix}} \longrightarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

pivot col

free col

Rank $= r = 1$.

$$C(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\right\}$$

(line)

$$N(A) = \text{span}\left\{\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right\}$$

(plane)

Note: $A = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -2 \end{pmatrix} = \vec{u}\,\vec{v}^{\,T}$

$A\vec{x} = 0 \Rightarrow \vec{u}\,(\vec{v}^{\,T}x) = 0$

Check: each vector in $N(A)$ is perpendicular to $\vec{v} = (1, 3, -2)$