

NULLSPACE OF A

Defⁿ The nullspace $N(A)$ consists of all solutions to $A\vec{x} = \vec{0}$.

If $A \in \mathbb{R}^{m \times n}$, the $N(A)$ is a subspace of \mathbb{R}^n .

Why? Suppose \vec{x}, \vec{y} are in the nullspace.

$$\text{Then } A\vec{x} = \vec{0} \text{ and } A\vec{y} = \vec{0}$$

$$\text{Using rules for matrix multiplication, } A(\underbrace{\vec{x} + \vec{y}}) = \vec{0}$$

Then $\vec{x} + \vec{y}$ is also in the nullspace.

$$\text{Similarly, } c(A\vec{x}) = c\vec{0}$$

$$\Rightarrow A(c\vec{x}) = \vec{0}$$

$\Rightarrow c\vec{x}$ is in the nullspace.

Linear combinations are in the nullspace. Therefore $N(A)$ is a subspace.

SOLVING $A\vec{x} = \vec{b}$ IN COMPLETE GENERALITY

Defⁿ A matrix is in reduced row echelon form (rref) if it is both

- (i) upper triangular and
- (ii) contains only a single non-zero entry - a '1' - in each pivot column.

Example

$$\begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix} \xrightarrow[\substack{-2 \text{ row (1)} \\ + \text{ row (2)}}]{\substack{\text{(elimination} \\ \text{top to bottom)}}} \begin{bmatrix} 1 & 1 & 3 \\ 0 & -1 & -6 \end{bmatrix} \xrightarrow[\text{row (1)}]{\substack{\text{(elimination} \\ \text{bottom to top)}}} \begin{bmatrix} 1 & 0 & -3 \\ 0 & -1 & -6 \end{bmatrix}$$

Upper triangular

row (2) \rightarrow -1 row (2)



$$\begin{bmatrix} \textcircled{1} & 0 & -3 \\ 0 & \textcircled{1} & 6 \end{bmatrix}$$

Reduced Row Echelon Form

$\uparrow \quad \uparrow$
pivot columns free column

Solving $A\vec{x} = \vec{0}$ when A is in reduced row echelon form

From our previous example, $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \end{bmatrix}$ and $R = \text{rref}(A) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 6 \end{bmatrix}$
 $A\vec{x} = \vec{0}$ and $R\vec{x} = \vec{0}$ have same solution

Solving $R\vec{x} = \vec{0}$, we have $\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$


Pivot cols. free col.

z is a 'free' variable

$$\left\{ \begin{array}{ll} x - 3z = 0 & \text{or} \quad x = 3z \\ y + 6z = 0 & \text{or} \quad y = -6z \end{array} \right.$$

How to choose z ?

$z = 0$ (trivial solution) $x = y = 0$

$z = 1$ special solution $x = 3; y = -6$

Another Example

Let $R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

↑ pivot col ↑ pivot col

(cols. with the first 1 in each row)

free cols

reduced row echelon form
- upper triangular
- each pivot col. has only 1 as non-zero entry

Solving $R\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2 Special solns

choose $y=1, w=0;$ we get $x=-2, z=0$
 $y=0, w=1;$ $x=-1, z=-1$

Since we have
2 "free" variables
or cols.

So

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \vec{0}$$

column 2 is a linear
combination of column 1

col. 4 is a linear
combination of
cols. 1 and 3

$$N(R) = N(A) = \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right)$$

Example: Give all possible solutions to $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 2 & 4 & 0 \\ 1 & 4 & 2 \\ 0 & 8 & 8 \end{bmatrix}$

Step 1 $[A : \vec{0}] \xrightarrow{\text{elimination}} \text{rref}(A)$

$$\begin{bmatrix} 2 & 4 & 0 & : & 0 \\ 1 & 4 & 2 & : & 0 \\ 0 & 8 & 8 & : & 0 \end{bmatrix} \xrightarrow{\substack{\text{row } ② \rightarrow \\ \text{row } ② - \frac{1}{2}\text{row } ①}} \begin{bmatrix} 2 & 4 & 0 & : & 0 \\ 0 & 2 & 2 & : & 0 \\ 0 & 8 & 8 & : & 0 \end{bmatrix} \xrightarrow{\substack{\text{row } ③ \rightarrow \\ \text{row } ③ - 4\text{row } ②}} \begin{bmatrix} 2 & 4 & 0 & : & 0 \\ 0 & 2 & 2 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$


divide
by pivots \rightarrow

$$\begin{bmatrix} 1 & 2 & 0 & : & 0 \\ 0 & 1 & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

Upper triangular
now perform elimination
bottom to top

$\xrightarrow{\substack{\text{row } ① \rightarrow \\ \text{row } ① - 2\text{row } ②}}$

$$\begin{bmatrix} \textcircled{1} & 0 & -2 & : & 0 \\ 0 & \textcircled{1} & 1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$



reduced row
echelon form $[R : \vec{0}]$ where $R = \text{rref}(A)$

$$[R:\vec{b}] = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solving $R\vec{x} = \vec{b}$ $x - 2z = 0$ where z is a "free" variable
 $y + z = 0$

Special solution obtained by setting $z=1$. Then $x=2$ and $y=-1$

Then $N(A) = N(R) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$

Note: $C(A) = \text{span of original pivot columns}$
 $= \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 8 \end{bmatrix} \right)$