REVIEW PROBLEMS

① Suppose $(x_1, x_2) + (y_1, y_2)$ is defined to be $(x_2 + y_1, x_1 + y_2)$. With the usual multiplication $c\vec{x} = (cx_1, cx_2)$, which of the eight conditions for a vector space are not satisfied?

$\boxed{\text{Sol}^n}$ Let's check axiom (i)

$$\vec{x} + \vec{y} = (x_1, x_2) + (y_1, y_2) = (\underline{x_2 + y_1}, x_1 + y_2)$$
$$\vec{y} + \vec{x} = (y_1, y_2) + (x_1, x_2) = (\underline{y_2 + x_1}, y_1 + x_2)$$

$$\vec{x} + \vec{y} \neq \vec{y} + \vec{x}$$

How about axiom (ii)?

Axiom (ii) fails

$$\vec{x} + (\vec{y} + \vec{z}) = (x_1, x_2) + (y_2 + z_1, y_1 + z_2) = (\underline{x_2 + y_2 + z_1}, x_1 + y_1 + z_2)$$
$$(\vec{x} + \vec{y}) + \vec{z} = (x_2 + y_1, x_1 + y_2) + (z_1, z_2) = (\underline{x_1 + y_2 + z_1}, x_2 + y_1 + z_2)$$

How about axiom (viii)?

Axiom (viii) fails

$$(c_1 + c_2)\vec{x} = (\underline{(c_1 + c_2)x_1}, (c_1 + c_2)x_2) \neq (\underline{c_1 x_2 + c_2 y_1}, c_1 x_1 + c_2 y_2)$$

② (#5 from Problem Set 3.1)

> **Note:** M is the vector space of real 2×2 matrices

(a) Describe a subspace of M that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$

$\boxed{\text{Sol}^n}$ Consider the set $M_1 = \left\{ \begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; c \in \mathbb{R} \right\}$

$A \in M_1$ but $B \notin M_1$

$M_1 \subseteq M$ and we can check that M_1 is closed under addition, scalar multiplication.

(b) If a subspace of M does contain A and B, must it contain I?

$\boxed{\text{Sol}^n}$ Yes! Why? To be a subspace, all linear combinations should be in the subspace. $\underbrace{A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$ is a linear combination of A, B and should therefore be in the subspace.

(c) Describe a subspace of M that contains no nonzero diagonal matrices

$\boxed{\text{Sol}^n}$ $M_2 = \left\{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \in \mathbb{R}^{2\times 2} \;\middle|\; a, b \in \mathbb{R} \right\}$ Check: $M_2 \subseteq M$; linear combinations stay in M_2

③ Which rule is broken if multiplying $f(x)$ by c gives the function $f(2cx)$? Let the definition of $f(x)+g(x)$ remain unchanged.

$\boxed{\text{Sol}^n}$

Here is a rule which breaks

Axiom (viii): $(c_1+c_2)\vec{x} = c_1\vec{x} + c_2\vec{x}$

In our problem, $(c_1+c_2)f(x) = f((2c_1+2c_2)x)$ —— ①

$c_1 f(x) + c_2 f(x) = f(2c_1 x) + f(2c_2 x)$ —— ②

Does ① = ② always for all f?

No! Here is an example: $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$

Then $f((2c_1+2c_2)x) = ((2c_1+2c_2)x)^2 = \underline{(4c_1^2 + 8c_1c_2 + 4c_2^2)x^2}$

$f(2c_1x) + f(2c_2x) = \underline{4c_1^2x^2 + 4c_2^2x^2}$

④ (This problem concerns subspaces)

(a) Find a set of vectors in $\mathbb{R}^2$ for which $\vec{x}+\vec{y}$ stays in the set, but $\sqrt{2}\,\vec{x}$ may be outside

$\boxed{\text{Sol}^n}$ Consider $A \subseteq \mathbb{R}^2$ with $A = \{ (x_1, x_2) \mid x_1, x_2 \in \underbrace{\mathbb{Q}}_{\text{rational numbers}} \}$

For $\vec{x}, \vec{y} \in A$, $(x_1, x_2) + (y_1, y_2) \in A$ since sum of rational numbers is still a rational number.

but consider $\vec{x} \in A$ with $\vec{x} = \left(\frac{1}{2}, \frac{1}{3}\right)$

$\sqrt{2}\,\vec{x} = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}\right)$. Since $\frac{\sqrt{2}}{2}$ and $\frac{\sqrt{2}}{3}$ are not rational, $\sqrt{2}\vec{x} \notin \mathbb{Q}$.

(b) Find a set of vectors in $\mathbb{R}^2$ for which every $c\vec{x}$ stays in the set but $\vec{x}+\vec{y}$ may be outside.

$\boxed{\text{Sol}^n}$ Consider $\mathbb{R}^2$ with the y-axis removed. (The x-axis and the origin remains)

⑤ Let P be the plane in $\mathbb{R}^3$ with equation $2x+y+z=2$.

Find two vectors in P such that their sum is not in P

$\boxed{\text{Sol}^n}$ <u>Note:</u> $(0,0,0) \notin P$. Hence P cannot be a subspace of $\mathbb{R}^3$.

Consider $\vec{a} = (1,0,0)$ and $\vec{b} = (0,2,0)$.

We see both $\vec{a}, \vec{b}$ are in the plane P.

However, $\vec{a}+\vec{b} = (1,2,0)$

$\vec{a}+\vec{b}$ is not in the plane since $2(1)+2+0 = 4 \neq 2$.

⑥ (#20 from Problem Set 3.1)

For which right sides (find a condition on b_1, b_2, b_3) are these systems solvable?

(a) $$\begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let's perform elimination on $[A \vdots \vec{b}]$

$$\left[\begin{array}{ccc:c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array}\right] \xrightarrow[\text{row ③} + \text{row ①}]{\text{row ②} - 2\,\text{row ①}} \left[\begin{array}{ccc:c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array}\right]$$

When do we have a solution?

$$\boxed{b_2 - 2b_1 = 0 \quad \text{and} \quad b_3 + b_1 = 0}$$

(b) $$\begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Elimination gives

$$\left[\begin{array}{cc:c} 1 & 4 & b_1 \\ 2 & 9 & b_2 \\ -1 & -4 & b_3 \end{array}\right] \xrightarrow[\text{row ③} + \text{row ①}]{\text{row ②} - 2\,\text{row ①}} \left[\begin{array}{cc:c} 1 & 4 & b_1 \\ 0 & 1 & b_2 - 2b_1 \\ 0 & 0 & b_3 + b_1 \end{array}\right]$$

Solution only if $\boxed{b_3 + b_1 = 0}$