Lecture 12

9/28/2016

Last Time:

* Transpose of a matrix
* Permutation matrices
* Elimination with row exchanges

$$\underbrace{P}_{\text{permutation matrix}} A = \underbrace{L}_{\text{lower triangular}} \underbrace{U}_{\text{upper triangular}}$$

Today

* Vector Spaces

VECTOR SPACES

Consider vectors in three-dimensional space

We know how to evaluate linear combinations using the definitions of

<u>Vector Addition</u> Let $\vec{p}, \vec{q} \in \mathbb{R}^3$ with $\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$, $\vec{q} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$

Then $\vec{p} + \vec{q} = \begin{pmatrix} p_1 + q_1 \\ p_2 + q_2 \\ p_3 + q_3 \end{pmatrix}$ and $(\vec{p} + \vec{q}) \in \mathbb{R}^3$

<u>Scalar Multiplication</u> Let $c \in \mathbb{R}$

Then $c\vec{p} = \begin{pmatrix} c\,p_1 \\ c\,p_2 \\ c\,p_3 \end{pmatrix}$ and $(c\vec{p}) \in \mathbb{R}^3$

Also, $\vec{p}, \vec{q}$ and the addition/scalar multiplication operations satisfy certain properties such as the commutative property

We say $\mathbb{R}^3$ is a vector space.

The "space" in the vector space $\mathbb{R}^3$ refers to the set of all possible vectors $\vec{v} \in \mathbb{R}^3$ (all possible vectors in three-dimensional space)

To be a vector space, we need

* the "vectors" (for example set of all $\vec{v} \in \mathbb{R}^3$)
* Definitions of "vector" addition and scalar multiplication
  (definitions of $\vec{p} + \vec{q}$ and $c\vec{p}$)
* Closure under vector addition and scalar multiplication
  ($\vec{p} + \vec{q}$, $c\vec{p}$ give vectors also in three-dimensional space)
* Satisfy axioms/properties such as the commutative law
  (see text or excerpt from class)

We can similarly define "vector" addition and scalar multiplication for

Matrices $A, B \in \mathbb{R}^{2\times 2}$

$c \in \mathbb{R}$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} (a_{11}+b_{11}) & (a_{12}+b_{12}) \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) \end{bmatrix}}_{A+B}$$

$$cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Here too, properties such as the distributive law hold. There is a "zero element" $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ ...

Since $A+B \in \mathbb{R}^{2\times 2}$ and $cA \in \mathbb{R}^{2\times 2}$, we say that the set of 2×2 matrices is <u>closed</u> under vector addition and scalar multiplication

We can similarly define "vector" addition and scalar multiplication for

Polynomials Let $\mathbb{P}_2$ be the set of all possible polynomials of degree 2 or less.

these can be written as $p(x) = a_0 + a_1 x + a_2 x^2$, $a_0, a_1, a_2 \in \mathbb{R}$

"Vector" Addition

$p, q \in \mathbb{P}_2$

$$p(x) = a_0 + a_1 x + a_2 x^2$$
$$q(x) = b_0 + b_1 x + b_2 x^2$$

$$(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$$

$(p+q) \in \mathbb{P}_2$ (still a quadratic)

Example:

$p(x) = 1 + 2x + 3x^2$

$q(x) = 2 + x - x^2$

$(p+q)(x) = 3 + 3x + 2x^2$

Scalar Multiplication

$c \in \mathbb{R}$

$$(cp)(x) = ca_0 + ca_1 x + ca_2 x^2$$

$cp \in \mathbb{P}_2$

Example:

$2p(x) = 2 + 4x + 6x^2$

Some other examples of "vectors"

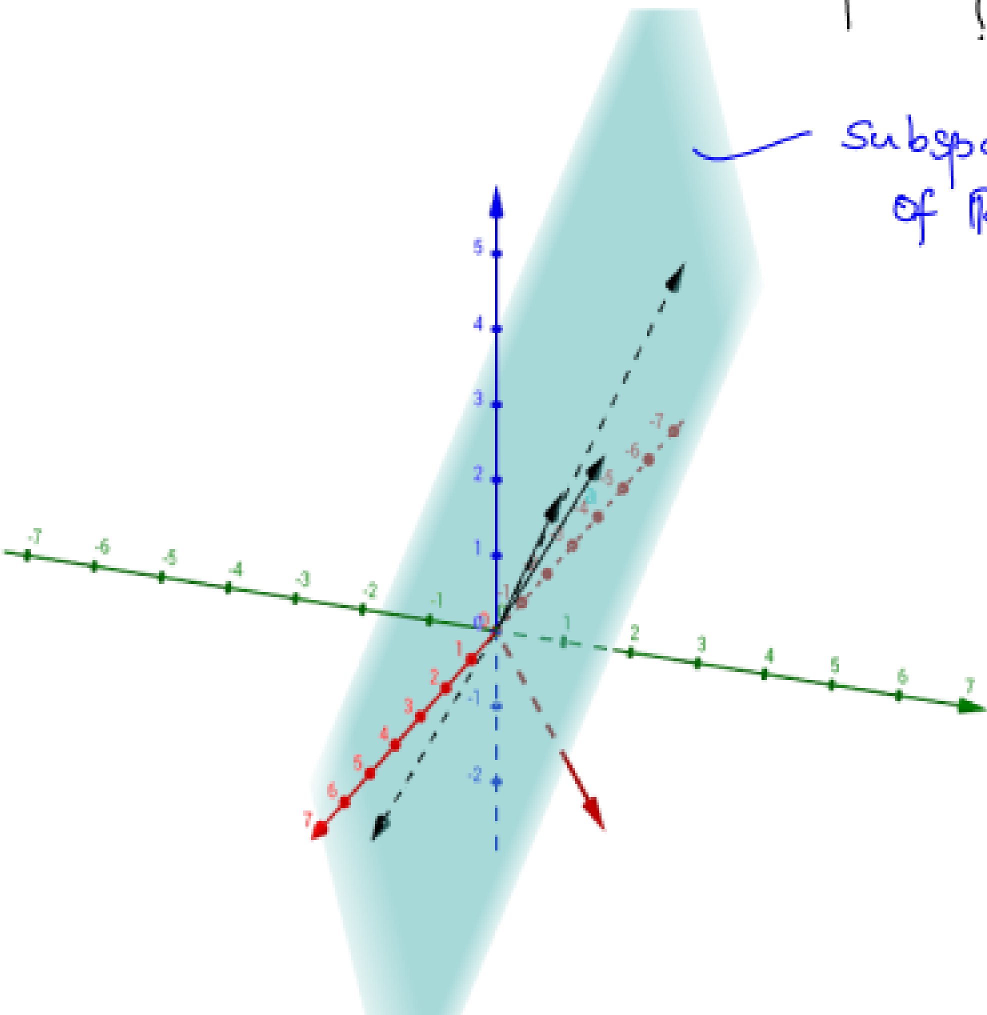
- $\mathbb{R}^n$ (vectors in n-dimensional space)
- $\mathbb{C}^n$ (vectors whose components are complex numbers)
- functions $f: \mathbb{R} \to \mathbb{R}$
- matrices $\mathbb{R}^{m \times n}$

Examples of Vector Spaces

- vector space of all real 2x2 matrices, M
- vector space of all real function $f(x)$, F
- vector space consisting of only the zero vector, Z

SUBSPACES

What if we are interested in only <u>some</u> (a subset of) vectors in three-dimensional space?



subspace of $\mathbb{R}^3$

<u>Example</u>

A plane through $(0,0,0)$ is a subspace of $\mathbb{R}^3$

$\boxed{\text{Def}^n}$ A subspace of a vector space is a set of vectors (including $\vec{0}$) that satisfies two requirements:

If $\vec{v}$ and $\vec{w}$ are vectors in the subspace and c is any scalar, then

(i) $\vec{v} + \vec{w}$ is in the subspace

(ii) $c\,\vec{v}$ is in the subspace

set of vectors is closed under addition and scalar multiplication

Examples of subspaces of $\mathbb{R}^3$

* Line through $(0,0,0)$

* Plane through $(0,0,0)$

* $\mathbb{R}^3$

* the vector $(0,0,0)$

Here are two subspaces of M - the vector space of all 2x2 matrices

$$L = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

set of all lower triangular matrices

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$

set of all diagonal matrices

$\boxed{\text{Def}^{n}}$ Given two vectors $\vec{u}, \vec{v}$ we define

$$\text{span}(\vec{u}, \vec{v}) = \text{the set of all possible linear combinations of } \vec{u}, \vec{v}$$

$$= \{ a\vec{u} + b\vec{v} \text{ for any scalars } a, b \}$$

The span of $\vec{u}_1, \vec{u}_2, \ldots\ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots\ldots, \vec{u}_n$

<u>Note:</u> If V is a vector space, then $\text{span}(\vec{u}, \vec{v}) \subseteq V$ for all $\vec{u}, \vec{v} \in V$

A subspace containing $\vec{u}, \vec{v}$ must contain $\text{span}(\vec{u}, \vec{v})$

COLUMN SPACE OF A

The column space of A contains all linear combinations of the columns of A

The column space of A is denoted as $C(A)$

Let $A \in \mathbb{R}^{m \times n}$. Using the function view of a matrix

$$\vec{x} \in \mathbb{R}^n \longrightarrow \boxed{A \in \mathbb{R}^{m \times n}} \longrightarrow \vec{y} \in \mathbb{R}^m \qquad \vec{y} = A\vec{x}$$

$$A\vec{x} = \begin{pmatrix} \underbrace{\begin{matrix} | \\ \vec{a}_1 \\ | \end{matrix}}_{\vec{a}_i \in \mathbb{R}^m} & \begin{matrix} | \\ \vec{a}_2 \\ | \end{matrix} & \cdots\cdots & \begin{matrix} | \\ \vec{a}_n \\ | \end{matrix} \end{pmatrix} \vec{x} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots\cdots + x_n \vec{a}_n$$

$$= \text{span}(\vec{a}_1, \ldots\ldots, \vec{a}_n)$$

This is a subspace of $\mathbb{R}^m$

Alternate notation <u>Range</u> of A $A\vec{x}$ always in $C(A)$

Why are we interested in $C(A)$?

The system $A\vec{x} = \vec{b}$ is solvable if and only if $\vec{b}$ is in the column space of A.

Example: $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$C(A) = \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

= plane containing the the columns $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

$C(A)$