Lecture 11

9/26/16

Last Time

* LU Factorization
* LDU Factorization

With no row exchanges

$$\underbrace{A = L}_{\text{lower triangle (multipliers } \ell_{ij} \text{ at appropriate positions)}} \underbrace{U}_{\text{upper triangular (pivots on diagonal)}}$$

Today

* Transpose
* Permutations

TRANSPOSE

Def^n Let $A \in \mathbb{R}^{m \times n}$. The <u>transpose</u> of A is $A^T \in \mathbb{R}^{n \times m}$ formed by exchanging rows with columns.

$$\boxed{(A^T)_{ij} = A_{ji}}$$

<u>Example:</u>

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 4 \end{bmatrix}$$

Column 1

entry 3, 2

row 1

entry 2, 3

PROPERTIES OF THE TRANSPOSE OF MATRICES

Sum $(A+B)^T = A^T + B^T$

Product $(AB)^T = B^T A^T$

(also extends to 3 or more factors
for example: $A = LDU$; $A^T = U^T D^T L^T$)

Inverse $(A^{-1})^T = (A^T)^{-1}$

Understanding the product rule

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & -4 \end{bmatrix}$$

col 1

$$0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -1 & -4 \end{bmatrix}$$

row 1

$$0\begin{bmatrix} 1 & -1 \end{bmatrix} + 2\begin{bmatrix} 2 & 3 \end{bmatrix}$$

PROPERTIES OF THE TRANSPOSE OF MATRICES

Sum $\quad (A+B)^T = A^T + B^T$

Product $\quad (AB)^T = B^T A^T$

Inverse $\quad (A^{-1})^T = (A^T)^{-1}$

Understanding the inverse rule

If A is invertible, $A^{-1}A = I$

$\Rightarrow (A^{-1}A)^T = I^T = I$ (taking the transpose)

$\Rightarrow A^T (A^{-1})^T = I$ (product rule)

$\Rightarrow (A^{-1})^T = (A^T)^{-1}$ (multiply by $(A^T)^{-1}$)

INNER AND OUTER PRODUCTS

(Recall) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We denoted the inner product by $\vec{x} \cdot \vec{y}$

Using transpose notation, we have

dot product or inner product is $\vec{x}^T \vec{y}$

$\vec{x}^T \in \mathbb{R}^{1 \times n}$ $\vec{y} \in \mathbb{R}^{n \times 1}$

$\vec{x}^T \vec{y} \in \mathbb{R}^{1 \times 1}$

<u>Note:</u> The outer product is $\vec{x}\, \vec{y}^T$

$\vec{x} \in \mathbb{R}^{n \times 1}$ $\vec{y}^T \in \mathbb{R}^{1 \times n}$

$\vec{x} \vec{y}^T \in \mathbb{R}^{n \times n}$

Symmetric Matrices

Def^n A matrix $S \in \mathbb{R}^{n \times n}$ is called symmetric if $S^T = S$

For example, consider the row exchange matrix

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{32}^T$$

Recall also that $P_{32}^{-1} = P_{32}$

A row exchange or swap matrix is always symmetric

Another example Consider any $A \in \mathbb{R}^{m \times n}$. Then

transpose of $\underbrace{A^T A}_{\in \mathbb{R}^{n \times n}}$ is $(A^T A)^T = A^T (A^T)^T = A^T A$

Transpose of $\underbrace{A A^T}_{\in \mathbb{R}^{m \times m}}$ is $(A A^T)^T = A A^T$

Symmetric Matrices in Elimination

$$\underbrace{\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_{U}$$

Note: $U = L^T$

If $S = S^T$ has factorization LDU with no row exchange, then $U = L^T$, or $\underbrace{S}_{\text{Symmetric matrix}} = LDL^T$

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (2x2 matrices)

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Identity matrix}} \qquad \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{swap matrix}}$$

2 total permutations

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (3x3 matrices)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

P_{32}: swap

P_{21}: swap

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

P_{31}: swap

$P_{32}P_{21}$: not swap, not symmetric, but product of swaps

6 total permutations

$6 = 3!$

$= (3)(2)(1)$

(row order) 123 132 213 231 312 321

$\boxed{\text{Thm}}$ Every permutation matrix P has $P^{-1} = P^{T}$

$\boxed{\text{Proof}}$ Let P be a permutation matrix.

Then $P = S_1 \cdots\cdots S_{n-1}$ ($n-1$ swaps

Hence,
$$
\begin{aligned}
P^{-1} &= (S_1 S_2 \cdots\cdots S_{n-1})^{-1} \\
&= S_{n-1}^{-1} \cdots\cdots S_2^{-1} S_1^{-1} && \left((AB)^{-1} = B^{-1}A^{-1} \right) \\
&= S_{n-1} \cdots\cdots S_2 S_1 && (\text{each swap matrix is its own inverse}) \\
&= S_{n-1}^{T} \cdots\cdots S_2^{T} S_1^{T} && (\text{each swap matrix is its own transpose}) \\
&= (S_1 S_2 \cdots\cdots S_{n-1})^{T} && \left((AB)^{T} = B^{T}A^{T} \right) \\
&= P^{T}
\end{aligned}
$$

PA = LU FACTORIZATION

upper triangular

Every matrix $A \in \mathbb{R}^{n \times n}$ has a PLDU factorization

permutation matrix — lower triangular — diagonal

<u>Example</u>

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Swap rows ② and ③

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

row ② − 2 row ①

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Scale with diagonal pivots

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad E_{21}P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$PA = LU$ FACTORIZATION

Every matrix $A \in \mathbb{R}^{n \times n}$ has a $PLDU$ factorization

permutation matrix

lower triangular

diagonal

upper triangular

<u>Example</u> $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$

$$P_{32}A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

Using $P_{32}^{-1} = P_{32}^{T} = P_{32}$, we have $A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$