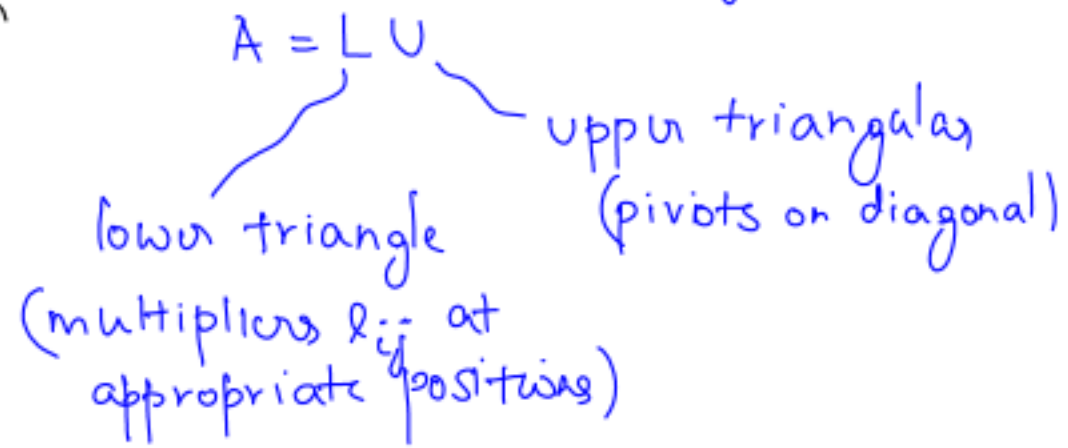


LAST TIME

- * LU Factorization
- * LDU Factorization

With no row exchanges



TODAY

- * Transpose
- * Permutations

TRANSPOSE

Defⁿ

Let $A \in \mathbb{R}^{m \times n}$. The transpose of A is $A^T \in \mathbb{R}^{n \times m}$ formed by exchanging rows with columns.

$$(A^T)_{ij} = A_{ji}$$

Example:

$$A = \begin{bmatrix} 1 & 7 \\ 2 & 0 \\ -1 & 4 \end{bmatrix}$$

Column 1

entry 3,2

$$A^T = \begin{bmatrix} 1 & 2 & -1 \\ 7 & 0 & 4 \end{bmatrix}$$

row 1

entry 2,3

PROPERTIES OF THE TRANSPOSE OF MATRICES

Sum $(A+B)^T = A^T + B^T$

Product $(AB)^T = B^T A^T$ (also extends to 3 or more factors
for example: $A = LDU$; $A^T = U^T D^T L^T$)

Inverse $(A^{-1})^T = (A^T)^{-1}$

Understanding the product rule

$$AB = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 6 & -4 \end{bmatrix}$$

col 1

$$0 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ -1 & -4 \end{bmatrix}$$

row 1

$$0 \begin{bmatrix} 1 & -1 \end{bmatrix} + 2 \begin{bmatrix} 2 & 3 \end{bmatrix}$$

PROPERTIES OF THE TRANSPOSE OF MATRICES

SUM

$$(A+B)^T = A^T + B^T$$

PRODUCT

$$(AB)^T = B^T A^T$$

INVERSE

$$(A^{-1})^T = (A^T)^{-1}$$

Understanding the inverse rule

If A is invertible, $A^{-1}A = I$

$$\Rightarrow (A^{-1}A)^T = I^T = I$$

(taking the transpose)

$$\Rightarrow A^T (A^{-1})^T = I$$

(product rule)

$$\Rightarrow (A^{-1})^T = (A^T)^{-1}$$

(multiply by $(A^T)^{-1}$)

INNER AND OUTER PRODUCTS

(Recall) Let $\vec{x}, \vec{y} \in \mathbb{R}^n$. We denoted the inner product by $\vec{x} \cdot \vec{y}$

Using transpose notation, we have

dot product or inner product is $\vec{x}^T \vec{y}$

$\vec{x}^T \in \mathbb{R}^{1 \times n}$ $\vec{y} \in \mathbb{R}^{n \times 1}$

$\vec{x}^T \vec{y} \in \mathbb{R}^{1 \times 1}$

The diagram shows the expression $\vec{x}^T \vec{y}$ at the top. Two blue curly braces extend downwards from \vec{x}^T and \vec{y} to their respective dimensions: $\vec{x}^T \in \mathbb{R}^{1 \times n}$ and $\vec{y} \in \mathbb{R}^{n \times 1}$. A third blue curly brace extends downwards from the entire expression $\vec{x}^T \vec{y}$ to its resulting dimension: $\vec{x}^T \vec{y} \in \mathbb{R}^{1 \times 1}$.

Note: The outer product is $\vec{x} \vec{y}^T$

$\vec{x} \in \mathbb{R}^{n \times 1}$ $\vec{y}^T \in \mathbb{R}^{1 \times n}$

$\vec{x} \vec{y}^T \in \mathbb{R}^{n \times n}$

The diagram shows the expression $\vec{x} \vec{y}^T$ at the top. Two blue curly braces extend downwards from \vec{x} and \vec{y}^T to their respective dimensions: $\vec{x} \in \mathbb{R}^{n \times 1}$ and $\vec{y}^T \in \mathbb{R}^{1 \times n}$. A third blue curly brace extends downwards from the entire expression $\vec{x} \vec{y}^T$ to its resulting dimension: $\vec{x} \vec{y}^T \in \mathbb{R}^{n \times n}$.

SYMMETRIC MATRICES

Defⁿ A matrix $S \in \mathbb{R}^{n \times n}$ is called symmetric if $S^T = S$

For example, consider the row exchange matrix

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = P_{32}^T$$

A row exchange or swap matrix is always symmetric

Recall also that $P_{32}^{-1} = P_{32}$

Another example Consider any $A \in \mathbb{R}^{m \times n}$. Then

transpose of $A^T A$ is $(A^T A)^T = A^T (A^T)^T = A^T A$
of $\in \mathbb{R}^{n \times n}$

Transpose of $A A^T$ is $(A A^T)^T = A A^T$
of $\in \mathbb{R}^{m \times m}$

Symmetric Matrices in Elimination

$$\begin{aligned} \underbrace{\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}}_A &= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix}}_U \\ &= \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}_U \end{aligned}$$

Note: $U = L^T$

If $S = S^T$ has factorization LDU with no row exchanges,

then $U = L^T$, or $S = \underbrace{LDL^T}$

Symmetric
matrix

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (2x2 matrices)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Identity
matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

swap
matrix

2 total
permutations

PERMUTATION MATRICES

A permutation matrix is the identity matrix with its rows reordered any way you want.

A row exchange or swap matrix is a special case

Examples: (3x3 matrices)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

swap

$$P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

swap

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

swap

$$P_{32} P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{21} P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

not swap
not symmetric
but product of
swaps

6 total
permutations

$$6 = 3! \\ = (3)(2)(1)$$

(row order) 1 2 3 1 3 2 2 1 3 2 3 1
3 1 2 3 2 1

Thm

Every permutation matrix P has $P^{-1} = P^T$

Proof

Let P be a permutation matrix.

Then $P = S_1 \cdots S_{n-1}$ ($n-1$ swaps)

$$\text{Hence, } P^{-1} = (S_1 S_2 \cdots S_{n-1})^{-1}$$

$$= S_{n-1}^{-1} \cdots S_2^{-1} S_1^{-1}$$

$$= S_{n-1} \cdots S_2 S_1$$

$$= S_{n-1}^T \cdots S_2^T S_1^T$$

$$= (S_1 S_2 \cdots S_{n-1})^T$$

$$= P^T$$

$$((AB)^{-1} = B^{-1} A^{-1})$$

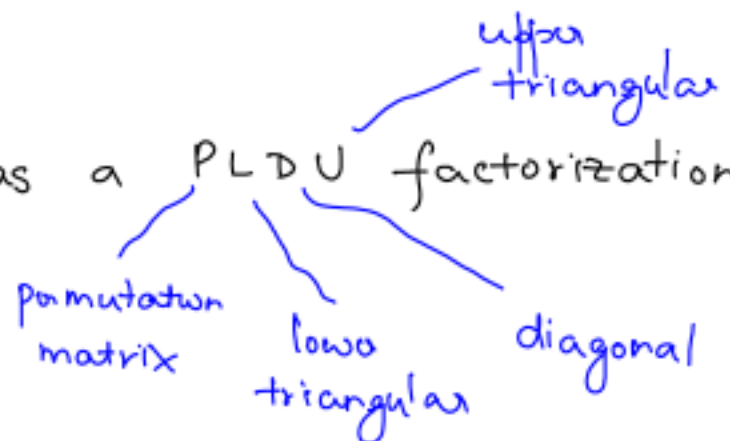
(each swap matrix is its own inverse)

(each swap matrix is its own transpose)

$$((AB)^T = B^T A^T)$$

PA = LU FACTORIZATION

Every matrix $A \in \mathbb{R}^{n \times n}$ has a PLDU factorization



Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

Swap rows
② and ③

$$P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

row ② - 2
row ①

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}P_{32}A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

scale with
diagonal pivots

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

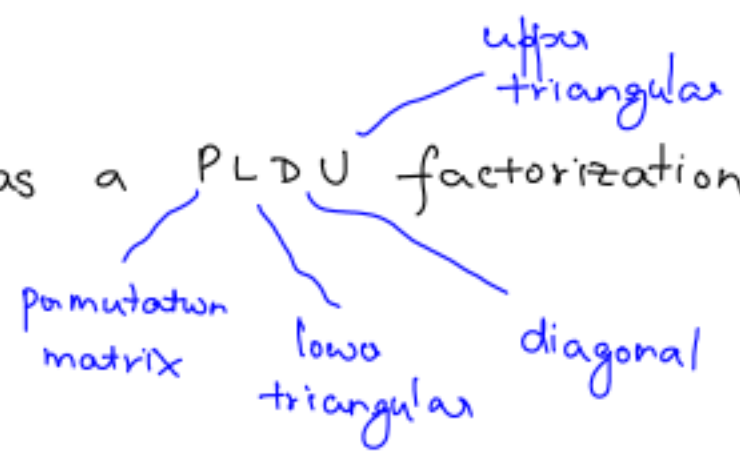
$$E_{21}P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{32}A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

PA = LU FACTORIZATION

Every matrix $A \in \mathbb{R}^{n \times n}$ has a PLDU factorization



Example

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 1 & 0 \end{bmatrix}$$

$$P_{32} A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_U$$

Using $P_{32}^{-1} = P_{32}^T = P_{32}$, we have

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_D \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_U$$