LECTURE 10

9/23/16

LAST TIME

* Inverse matrices $AA^{-1} = A^{-1}A = I$
* A is invertible if and only if it has n pivots
* $(AB)^{-1} = B^{-1}A^{-1}$
* Gauss-Jordan method solves $AA^{-1} = I$ $[A \,\vdots\, I] \xrightarrow[\text{reduced}]{\text{row}} [I \,\vdots\, A^{-1}]$

TODAY

* LU factorization

Matrix Factorization

Original matrix A becomes product of two or three special matrices

<u>TODAY</u> LU factorization

- arises from elimination
- L and U are triangular matrices

Review - Elimination

Let's perform elimination / row reduction on $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 1 & 1 & 2 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix} \qquad E_{31}E_{21}A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{3} & 1 \end{bmatrix} \qquad E_{32}E_{31}E_{21}A = \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}}_{U}$$

pivots

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

upper triangular form

With no row exchanges,

$$(E_{32} E_{31} E_{21}) A = U$$

Move over the E's to other side

$$A = (E_{32} E_{31} E_{21})^{-1} U$$

$$= \underbrace{(E_{21}^{-1} E_{31}^{-1} E_{32}^{-1})}_{L} U$$

$$= L U$$

We know that

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{21}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} \quad E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1/3 & 1 \end{bmatrix} \quad E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/3 & 1 \end{bmatrix}$$

lower triangular

$$E_{21}^{-1} E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix}$$

$$E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{1/2} & 1 & 0 \\ \boxed{1/2} & \boxed{1/3} & 1 \end{bmatrix}$$

$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}$ is lower triangular

* diagonal contains 1's
* multipliers unchanged

$\boxed{\text{LU Factorization}}$

$$\boxed{A = LU}$$

Elimination without row exchanges

Upper triangular U has pivots on its diagonal

Lower triangular L has 1's on its diagonal

The multipliers ℓ_{ij} are below the diagonal of L

LDU Factorization

Consider the coefficient matrix $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

$$E_{21} = \begin{bmatrix} 1 & 0 \\ -5/3 & 1 \end{bmatrix} \qquad E_{21}A = \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}}_{U}$$

$$\Rightarrow A = E_{21}^{-1} \begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix}}_{U}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 \\ 5/3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}}_{\breve{U}}$$

Split U into $D\breve{U}$

$$\begin{bmatrix} 3 & 4 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 4/3 \\ 0 & 1 \end{bmatrix}$$

pivots

Notes

(I) Why LU? This is how computers solve linear systems.

- factor A into L and U
- Solve

$A\vec{x} = \vec{b} \iff (LU)\vec{x} = \vec{b}$

$L(\underbrace{U\vec{x}}_{\vec{c}}) = \vec{b}$

(I) $L\vec{c} = \vec{b}$ solve by forward substitution

(II) $U\vec{x} = \vec{c}$ solve by back substitution

If A has all non-zero entries (we say A is a full matrix)

(II) we need about $\frac{1}{3}n^3$ multiplications and subtractions to do elimination

If A has many zero entries (we say A is a sparse matrix) elimination costs less.

(III) When a row of A starts with zeros, so does that row of L
When a column of A starts with zeros, so does that column of U

Problems

Compute the factorization $A = LU$ for the systems

① $$\begin{aligned} x + y &= 5 \\ x + 2y &= 7 \end{aligned}$$

② $$\begin{aligned} x + y + z &= 5 \\ x + 2y + 3z &= 7 \\ x + 3y + 6z &= 11 \end{aligned}$$