LECTURE 9

9/21/16

LAST TIME

* Matrix operations
* Interpretations of matrix multiplication
* Block matrices

TODAY

* Inverse Matrices

THE INVERSE

<u>Recall:</u> $B \in \mathbb{R}^{n \times n}$ is the inverse of $A \in \mathbb{R}^{n \times n}$ if

$$BA\vec{x} = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n$$

$$= I\vec{x} \quad \text{where } I \text{ is the } n \times n \text{ identity matrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & \\ 0 & 1 & 0 & & 0 \\ & 0 & 1 & \ddots & \\ & 0 & & 0 & 1 & 0 \\ & & & & 0 & 1 \end{bmatrix}$$

Hence, $B \in \mathbb{R}^{n \times n}$ is the inverse of $A \in \mathbb{R}^{n \times n}$ if and only if $BA = I$.

<u>Note:</u> Although $BA \neq AB$ in general, if $BA = I$, then $AB = I$ too!

Uniqueness of the Inverse

Can $A \in \mathbb{R}^{n \times n}$ have two inverses? $\boxed{\text{No}}$

Why? If $BA = I = CA$,

then $\underline{AB = I}$ too.

So, $CA = I$

$\Rightarrow (CA)B = B$ (multiplying by B from the right)

$\Rightarrow C(AB) = B$ (associative law)

$\Rightarrow C = B.$ (since $AB = I$)

A can have only <u>one inverse</u> and we will denote it as A^{-1}

Notes on Inverses

Note 1 The inverse exists if and only if elimination produces n pivots

Note 2 If A is invertible, the one and only solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$

Why? $A\vec{x} = \vec{b} \xrightarrow[\text{by } A^{-1}]{\text{multiply}} A^{-1}(A\vec{x}) = A^{-1}\vec{b}$ or $(A^{-1}A)\vec{x} = A^{-1}\vec{b}$

Note 3 Suppose there is a non-zero vector $\vec{x}$ such that $A\vec{x} = \vec{0}$.

Then A cannot have an inverse.

Note 4 If A is diagonal, $A = \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & 1/d_2 & \\ & & \ddots & \\ 0 & & & 1/d_n \end{bmatrix}$

Note 5 If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$

Why? $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = I$

FINDING THE INVERSE

We know that $A^{-1}A = AA^{-1} = I$

Hence $$A \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1^{-1} & \vec{a}_2^{-1} & \cdots\cdots & \vec{a}_n^{-1} \\ | & | & & | \end{bmatrix}}_{\text{column vectors of } A^{-1}} = I$$

Therefore $$\begin{bmatrix} | & | & & | \\ A\vec{a}_1^{-1} & A\vec{a}_2^{-1} & \cdots\cdots & A\vec{a}_n^{-1} \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & \cdots\cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix}$$

This is equivalent to the n matrix-vector equations

$$A\vec{a}_1^{-1} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\text{Columns of } I}, \quad A\vec{a}_2^{-1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \cdots\cdots, \ A\vec{a}_n^{-1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\text{Def}^n}$ Let $\vec{e}_j \in \mathbb{R}^n$ be the vector $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ j^{th} entry is a 1, the rest are 0

$A^{-1}A = I$ if and only if $A\vec{a}_j^{-1} = \vec{e}_j$ for all $j = 1, 2, \ldots, n$

j^{th} column of A^{-1}

$\rightsquigarrow$ Therefore, we can solve for A^{-1} by performing elimination n times

$A\vec{x} = \vec{e}_j$ has solution $\vec{x} = \vec{a}_j^{-1}$

$\left.[A \,\vdots\, \vec{e}_j] \longrightarrow \begin{array}{c}\text{make} \\ \text{upper triangular} \\ \text{and back substitute}\end{array}\right\} n \text{ times}$

Faster Method When making the augmented matrix upper triangular, our work only depends on A.

Therefore, we can solve for all the equations at once!

Example: Find the inverse of $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 2 & -2 \end{bmatrix}$ using elimination

$\boxed{\text{Step 1}}$ Construct augmented matrix

$[A \ I]$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{array}$$

$\boxed{\text{Step 2}}$ Perform elimination

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

Gaussian elimination

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \quad \text{row ② - 2 row ①}$$

finish with back substitution

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right] \quad \text{row ③ + row ②}$$

$\boxed{\text{Step 3}}$ Perform back substitution bottom to top; want zeros above pivots too

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & -4 & 2 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right]$$ row 2 + row 3

Gauss - Jordan Method

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & -2 & 0 & -4 & 2 & 1 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array}\right]$$ row 1 + row 2

Reduced echelon form

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2 & 1 \\ 0 & 1 & 0 & 2 & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right]$$ Divide each row by pivot

A^{-1}

Another Interpretation

$$\underbrace{[A \vdots I]}_{\text{augmented matrix}} \xrightarrow{\text{elimination}} D^{-1} \underbrace{E_{q} E_{q-1} \cdots\cdots P \cdots\cdots E_1}_{\text{elimination and/or row exchange matrices}} [A \vdots I] = [I \vdots A^{-1}]$$

division by pivots

Gauss Jordan

If $E = D^{-1} E_{q} E_{q-1} \cdots\cdots P \cdots\cdots E_1$, then $E[A \vdots I] = [EA \vdots EI] = [I \vdots A^{-1}]$

Therefore A^{-1} is a product of swap/elimination matrices

$\boxed{\text{Singular vs Invertible}}$

A^{-1} exists (and Gauss-Jordan finds it) exactly when A has n pivots

n pivots — invertible

$< n$ pivots — singular