LECTURE 7

9/16/16

LAST TIME

* Elimination using matrices
  - Elimination matrix E_{ij}
  - Permutation matrix P_{ij}

TODAY

* More exercises on elimination using matrices

REVIEW

Consider the following linear system of equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 3x_1 + 2x_2 + x_3 &= 7 \\ 2x_1 + x_2 + 2x_3 &= 1 \end{aligned} \xrightarrow[\text{Equation}]{\text{Matrix Vector}} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 7 \\ 1 \end{bmatrix}}_{\vec{b}}$$

Augmented matrix is $[A \ \vec{b}]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right]$$

Ready, Set, Eliminate!

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ \boxed{3} & 2 & 1 & 7 \\ \boxed{2} & 1 & 2 & 1 \end{array}\right]$$

Step 1

Want to zero out last two entries of Column 1.

How to eliminate $a_{21} = 3$?

Subtract 3 row ① from row ②

Let's construct the elimination matrix E_{21}

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21}[A\ b] = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 7 \\ 2 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ \boxed{2} & 1 & 2 & 1 \end{array}\right]$$

Next, we want to eliminate $a_{31} = 2$.

Construct elimination matrix $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$$E_{31}(E_{21}[A\ b]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ \boxed{2} & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & \boxed{-3} & -4 & -1 \end{array}\right]$$

next, we want to eliminate this entry

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & \boxed{-3} & -4 & -1 \end{array}\right]$$

Step 2

how do we eliminate the -3 in the last row?

- subtract $\left(\frac{3}{4}\right)$ row 2 from row 3

To eliminate $a_{32} = -3$, use the elimination matrix $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}$

$$E_{32}\left(E_{31}E_{21}\begin{bmatrix} A & b \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{3}{4} & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & -3 & -4 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -4 & -8 & 4 \\ 0 & 0 & 2 & -4 \end{array}\right]$$

Now solve by back-substitution: $\boxed{x_3 = -2}$

$-4x_2 + 16 = 4$ or $\boxed{x_2 = 3}$

$x_1 + 6 - 6 = 1$ or $\boxed{x_1 = 1}$

Examples of Elimination and Permutation Matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

subtracts 2 row 1 from row 2

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$$

adds 3 row 2 to row 3

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

exchanges row 2 and row 3

Problem Set

① Write down the augmented matrix $[A\ \vec{b}]$ for the linear system

$$x + 2y - z = -1$$
$$2x + 2y + z = 1$$
$$3x + 5y - 2z = -1$$

Which three elimination matrices are required to reduce the system to triangular form? Solve the system after reducing it to triangular form.

⑤ Solve the following system using elimination.

$$x_1 + 2x_2 - x_3 + 3x_4 = 2$$
$$2x_1 + 4x_2 - x_3 + 6x_4 = 5$$
$$x_2 \qquad + 2x_4 = 3$$

② Suppose E subtracts 7 times row ① from row ③.

(a) To invert that step you should ____ 7 times row ____ to row ____ .

(b) What inverse matrix E^{-1} takes that reverse step (so $E^{-1}E = I$)?

(c) If the reverse step is applied first (and then E) show that $EE^{-1} = I$.

③ Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ____ . If you change a_{33} to ____ , there is no third pivot.

④ Apply elimination to the 3 by 4 augmented matrix $[A\ \vec{b}]$. How do you know this system has no solution? Change the last number 6 so there is a solution.

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

1 $\boxed{\text{Solution sketch}}$

Augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right]$

$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ and $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}$ are the required elimination matrices

$$E_{21}[A \ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 3 & 5 & -2 & -1 \end{array}\right]$$

$$E_{31}E_{21}[A \ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 3 & 5 & -2 & -1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right]$$

$$E_{32}E_{31}E_{21}[A \ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{array}\right]$$

Back substitute to find $\boxed{z = -1, \ y = -3 \text{ and } x = 4}$

② Suppose E subtracts 7 times row ① from row ③

(a) To invert that step you should <u>add</u> 7 times row <u>1</u> to row <u>3</u>

(b) What "inverse matrix" E^{-1} takes that reverse step (so $E^{-1}E = I$)

(c) If the reverse step is applied first (and then E) show that $EE^{-1} = I$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -7 & 0 & 1 \end{bmatrix} . \qquad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 7 & 0 & 1 \end{bmatrix}$$

<u>Note:</u> to find E^{-1}, use multiplier -7 instead of 7 (which was used in E)

$$E \underbrace{E^{-1}}_{\text{applied first}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -7 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 7 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

③ Suppose $a_{33} = 7$ and the third pivot is 5. If you change a_{33} to 11, the third pivot is ______. If you change a_{33} to ______, there is no pivot.

Let the matrix be $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 7 \end{bmatrix}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow[\text{of elimination}]{\text{after some steps}} \begin{bmatrix} x & x & x \\ 0 & x & p \\ 0 & x & 7 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{after}} \begin{bmatrix} x & x & x \\ 0 & x & p \\ 0 & 0 & 5 \end{bmatrix}$$

multiplier

Hence $-\ell p + 7 = 5$ or $-\ell p = -2$.

Now, if $a_{33} = 11$, we have $-\ell p + 11 = -2 + 11 = \boxed{9}$ and

$-\ell p + a_{33} = 0$ if $a_{33} = -(-\ell p) = \boxed{2}$

④ Apply elimination to the 3 by 4 augmented matrix $[A\ \vec{b}]$. How do you know this system has no solution? Change the last number 6 so there is a solution

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

Solution Sketch

Augmented matrix is $[A\ \vec{b}] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6 \end{array}\right] \begin{array}{c} \\ -2R_1 \\ -3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -2 & 3 \end{array}\right] \begin{array}{c} \\ \\ -R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

non-zero hence no Solution

If last component of $\vec{b}$ was changed from 6 to $\boxed{3}$ then the system has (an infinite number of) solutions.

5 Solution Sketch

augmented matrix for problem

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \xrightarrow{-2R_1} \sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{array}\right) \; R_2 \Leftrightarrow R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

Now back-substitute $\boxed{x_3 = 1}$

$\boxed{x_4 = \text{free}}$

$\boxed{x_2 = 3 - 2x_4}$

$x_1 = 2 - 3x_4 + x_3 - 2x_2$ or $\boxed{x_1 = -3 + x_4}$

$\boxed{\text{Note:}}$ this system has infinitely many solutions. (3 equations, 4 unknowns)

Here is one : $\boxed{x_4 = 1;\ x_3 = 1;\ x_2 = 1;\ x_1 = -2}$