LECTURE 6

09/14/16

LAST TIME

* Idea of elimination
  - pivots and multipliers
* When does elimination break down?

TODAY

* Elimination using matrices

Review of Matrix Multiplication

Matrix - vector multiplication

$\boxed{\text{Linear combination of columns}}$

Let $A \in \mathbb{R}^{n \times m}$, $\vec{x} \in \mathbb{R}^m$

$$A\vec{x} = \underbrace{\begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ | & | & & | \end{bmatrix}}_{n \text{ rows, } m \text{ cols}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} | \\ \vec{a}_1 \\ | \end{bmatrix}}_{\substack{\vec{a}_1 \in \mathbb{R}^n \\ (n\text{-dimensional} \\ \text{space})}} + x_2 \begin{bmatrix} | \\ \vec{a}_2 \\ | \end{bmatrix} + \cdots + x_m \begin{bmatrix} | \\ \vec{a}_m \\ | \end{bmatrix}$$

$$= \sum_{i=1}^{m} x_j \vec{a}_j$$

Note: the product $(A\vec{x}) \in \mathbb{R}^n$

Equivalently,

$\boxed{\text{Dot product with rows}}$

$$A\vec{x} = \begin{bmatrix} - \text{ row } 1 - \\ - \text{ row } 2 - \\ \vdots \\ - \text{ row } n - \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} \text{row } 1 \cdot \vec{x} \\ \text{row } 2 \cdot \vec{x} \\ \vdots \\ \text{row } n \cdot \vec{x} \end{bmatrix}}_{\substack{\text{result is a vector} \\ \text{in } n\text{-dimensional space}}} \in \mathbb{R}^n$$

each row $\in \mathbb{R}^m$
(a vector in m-dimensional space)

Note: For matrix multiplication, number of columns in A must be equal to the dimension of $\vec{x}$.

Matrix - Matrix Multiplication

Let $A \in \mathbb{R}^{n \times m}$ and let $B \in \mathbb{R}^{m \times q}$

$$AB = A \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_q \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_q \\ | & | & & | \end{bmatrix}$$

each column vector is m-dimensional ($\in \mathbb{R}^n$)

each column $\in \mathbb{R}^n$

Example:

Let $A = \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & -1 \end{bmatrix}}_{4 \text{ cols}} \Big\} \text{3 rows}$ $\qquad B = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}}_{2 \text{ cols}} \Big\} \text{4 rows}$

$A \in \mathbb{R}^{3 \times 4}$ $\qquad B \in \mathbb{R}^{4 \times 2}$

$$AB = \left[\underbrace{A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{first col}} \quad \underbrace{A \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}_{\text{second col}} \right] =$$

$$\left[\underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{\text{first col of result}} \quad \underbrace{-\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{\text{second col of result}} \right] = \underbrace{\begin{bmatrix} 2 & -2 \\ 2 & 0 \\ 4 & -2 \end{bmatrix}}_{2 \text{ Cols}} \Big\} \text{3 rows} \in \mathbb{R}^{3 \times 2}$$

Review of Elimination

Consider the system of equations

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 3x_2 + 5x_3 &= 1 \\
6x_1 + 5x_2 + 5x_3 &= -3
\end{aligned}
$$

With elimination, we reduce the given matrix equation to $\boxed{\text{upper triangular}}$ form, which can then be solved by $\boxed{\text{back substitution}}$

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 3x_2 + 5x_3 &= 1 \\
6x_1 + 5x_2 + 5x_3 &= -3
\end{aligned}
\xrightarrow[\text{row ③} - \boxed{3}\times\text{row ①}]{\text{row ②} - \boxed{2}\times\text{row ①}}
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
x_2 - x_3 &= -1 \\
2x_2 - 4x_3 &= -6
\end{aligned}
\xrightarrow[\boxed{2}\times\text{row ②}]{\text{row ③} -}
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
1x_2 - x_3 &= -1 \\
-2x_3 &= -4
\end{aligned}
$$

multiplier

pivots

triangular

$\boxed{\text{Review of Elimination}}$

Consider the system of equations

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
4x_1 + 3x_2 + 5x_3 &= 1 \\
6x_1 + 5x_2 + 5x_3 &= -3
\end{aligned}
$$

With elimination, we reduce the given matrix equation to $\boxed{\text{upper triangular}}$ form, which can then be solved by $\boxed{\text{back substitution}}$

Pivots

$$
\begin{aligned}
2x_1 + x_2 + 3x_3 &= 1 \\
1x_2 - x_3 &= -1 \\
-2x_3 &= -4
\end{aligned}
\quad \xrightarrow[\text{Substitution}]{\text{Back}} \quad
\boxed{\begin{aligned} x_3 &= 2 \\ x_2 &= 1 \\ x_1 &= -3 \end{aligned}}
$$

Elimination in terms of Matrices

$$\left.\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}\right\} \text{ Given system of equations}$$

Let us start by writing this as a matrix equation $A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}}_{\vec{b}}$$

Now recall that the steps of elimination work on both the LHS and RHS of the given system of equation (i.e., both A and $\vec{b}$)

The Augmented Matrix

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$$\boxed{\text{Augmented matrix } \quad [A \ \ \vec{b}] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right]}$$

- the augmented matrix has $\vec{b}$ as an extra column.

Let us perform elimination on the augmented matrix.

During elimination, we only use two types of operations.

$\boxed{\text{Operation 1}}$ Subtract $l \times$ row j from row k

Here $l \in \mathbb{R}$ is the multiplier and $j < k$.

$\boxed{\text{Operation 2}}$ Switch two rows $j \longleftrightarrow k$

Both of these operations can be performed using matrices!

The Elimination Matrix

Can we construct a matrix E which performs the following:

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned} \xrightarrow[\text{row (2)} - \boxed{2} \times \text{row (1)}]{} \begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ x_2 - x_3 &= -1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}$$

For example, we want

$$\vec{b} = \begin{bmatrix} 1 \\ \boxed{1} \\ -3 \end{bmatrix} \xrightarrow[E]{?} \vec{b}_{new} = \begin{bmatrix} 1 \\ \boxed{-1} \\ -3 \end{bmatrix}$$

This can be performed using matrix multiplication with an "elimination" matrix

$$\underbrace{\begin{bmatrix} 1 \\ \boxed{-1} \\ -3 \end{bmatrix}}_{\vec{b}_{new}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 1 \\ \boxed{1} \\ -3 \end{bmatrix}}_{\vec{b}}$$

In general,

to construct an $\underbrace{\text{elimination matrix}}_{\text{sometimes called elementary matrix}}$ $\boxed{E_{ij}}$ which <u>subtracts</u> a multiple of $\boxed{\text{row } j \text{ from row } i}$

- Start with the identity matrix (which has 1's on the diagonal and 0's otherwise)
- Include an extra non-zero entry $\boxed{-\ell}$ in the $\boxed{i,j \text{ position}}$

multiplier in elimination step

Another example

To subtract $-\ell$ row ① from row ②, we use the elimination matrix

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\ell & 0 & 1 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 1 \\ -1 \\ \boxed{-3} \end{bmatrix} \xrightarrow[\text{with } \ell = 3]{E_{31}} \vec{b}_{new} = \begin{bmatrix} 1 \\ -1 \\ \boxed{-6} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}}_{\vec{b}} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -6 \end{bmatrix}}_{\vec{b}_{new}}$$

Recall that

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I}\underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{\vec{b}}=\underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{\vec{b}}$$

Identity matrix

$$\begin{array}{rl} & \text{col } ① \\ \text{row } ③ & \underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-\ell&0&1\end{bmatrix}}_{E_{31}}\underbrace{\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}}_{\vec{b}}=\underbrace{\begin{bmatrix}b_1\\b_2\\-\ell b_1+b_3\end{bmatrix}}_{\vec{b}_{new}}\end{array}$$

Elimination matrix

Here E_{31} subtracts ℓ times the first component from the third component

col ①

row ③

Note: the non-zero entries of E are all at or below the diagonal

Note: The elimination matrix acts|operates not only on vectors, but on matrices too!

Recall, our original problem

$$\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}}_{\vec{b}}$$

Here is the elimination matrix E_{21} with $\ell = 2$ acting on A

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{21} A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 6 & 5 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & -1 \\ 6 & 5 & 5 \end{bmatrix}$$

E_{21} subtracts $2 \times$ row 1 from row 2

Note: row 1 and row 3 are unchanged!

The first step of elimination can then be written as the elimination matrix E_{21} acting on the augmented matrix $[A \ \vec{b}]$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\substack{\text{Elimination} \\ \text{matrix} \\ E_{21}}} \underbrace{\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right]}_{\substack{[A \ \vec{b}] \\ \text{augmented matrix}}} = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 6 & 5 & 5 & -3 \end{array}\right]$$

Subtracts
2 × row 1
from row 2

Elimination

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned} \xrightarrow[\text{row } 3 - \boxed{3} \times \text{row } 1]{\text{row } 2 - \boxed{2} \times \text{row } 1} \begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ x_2 - x_3 &= -1 \\ 2x_2 - 4x_3 &= -6 \end{aligned} \xrightarrow{\text{row } 3 - \boxed{2} \times \text{row } 2} \begin{aligned} 2x_1 + x_2 + 3x_3 &= 1 \\ 1x_2 - x_3 &= -1 \\ -2x_3 &= -4 \end{aligned}$$

multiplier

pivots

triangular

Elimination using Matrices

$$E_{21}[A\ \vec{b}] = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 6 & 5 & 5 & -3 \end{array}\right]$$

$$E_{31}(E_{21}[A\ \vec{b}]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{-3} & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 6 & 5 & 5 & -3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right]$$

$$E_{32}(E_{31}E_{21}[A\ \vec{b}]) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-2} & 1 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -4 & -6 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

Caution

$EA \neq AE$ in most cases

- $EA \longrightarrow E$ acts on the rows of A (row operations)
- $AE \longrightarrow E$ acts on the columns of A (column operations)

Note:

For matrix multiplication

Associative law holds $\quad A(BC) = (AB)C$

Commutative law is false $\quad$ (usually) $AB \neq BA$

The Row Exchange Matrix

Recall that a row exchange was needed when elimination breaks down temporarily due to a 0 in the pivot

To exchange, for example, row 2 and row 3, we use a permutation or row-exchange matrix of the form

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Here is the row-exchange matrix in action

... acting on a vector

$$P_{23}\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ \boxed{3} \\ \boxed{-4} \end{bmatrix}}_{\vec{b}} = \underbrace{\begin{bmatrix} 1 \\ \boxed{-4} \\ \boxed{3} \end{bmatrix}}_{\vec{b}_{new}}$$

... acting on an augmented matrix

$$P_{23}\left[A \ \ \vec{b}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 4 & 10 & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 4 & 10 & 1 & 2 \end{array}\right]$$

In general,

The Row Exchange matrix $\boxed{P_{ij}}$ is the identity matrix with rows i and j reversed. When P_{ij} multiplies a matrix, it exchanges rows i and j.

Here is the row-exchange matrix which exchanges rows 2 and 4

$$P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Recall:

Here is the identity matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

① Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32} E_{31} E_{21} A = U.$$

Multiply those E's to get one matrix M that does elimination $MA = U$

to eliminate $a_{21} = 4$, we want -4 row ① + row ② $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

to eliminate $a_{31} = -2$, we do $-(-2)$ row ① + row ③ $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

Note.

$$E_{21} A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31} E_{21} A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix}$$

→ to make A triangular we need to subtract $2 \times$ row② from row③

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Multiply those E's to get one matrix M that does elimination $MA = U$.

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix}$$

$$E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \longrightarrow$$

to make A triangular we need to subtract 2 × row② from row③

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

① Which three matrices E_{21}, E_{31}, E_{32} put A into triangular form U?

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 6 & 1 \\ -2 & 2 & 0 \end{bmatrix} \quad \text{and} \quad E_{32}E_{31}E_{21}A = U.$$

Multiply those E's to get one matrix M that does elimination $MA = U$

$$E_{32}E_{31}E_{21} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}}_{E_{32}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{E_{31}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_{21}}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}}_{E_{32}E_{31}} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}}_{M}$$