LECTURE 05

9/12/16

LAST TIME:

* Linear Equations
* Geometry of linear equations
  - Row picture, column picture
* When do linear equations have
  (i) a single solution (ii) no solution (iii) many solutions

TODAY:

* Solving linear equations using ELIMINATION

The Idea of Elimination

How would you solve

$$x + 2y = 3 \quad \text{——— (a)}$$
$$3x - y = 2 \quad \text{——— (b)}$$

Step 1: Eliminate the unknown x from equation (b)

$$x + 2y = 3 \quad \text{(leave first equation unchanged)} \quad \text{——— (c)}$$
$$0x - 7y = -7 \quad \text{((b) - 3(a))} \quad \text{——— (d)}$$

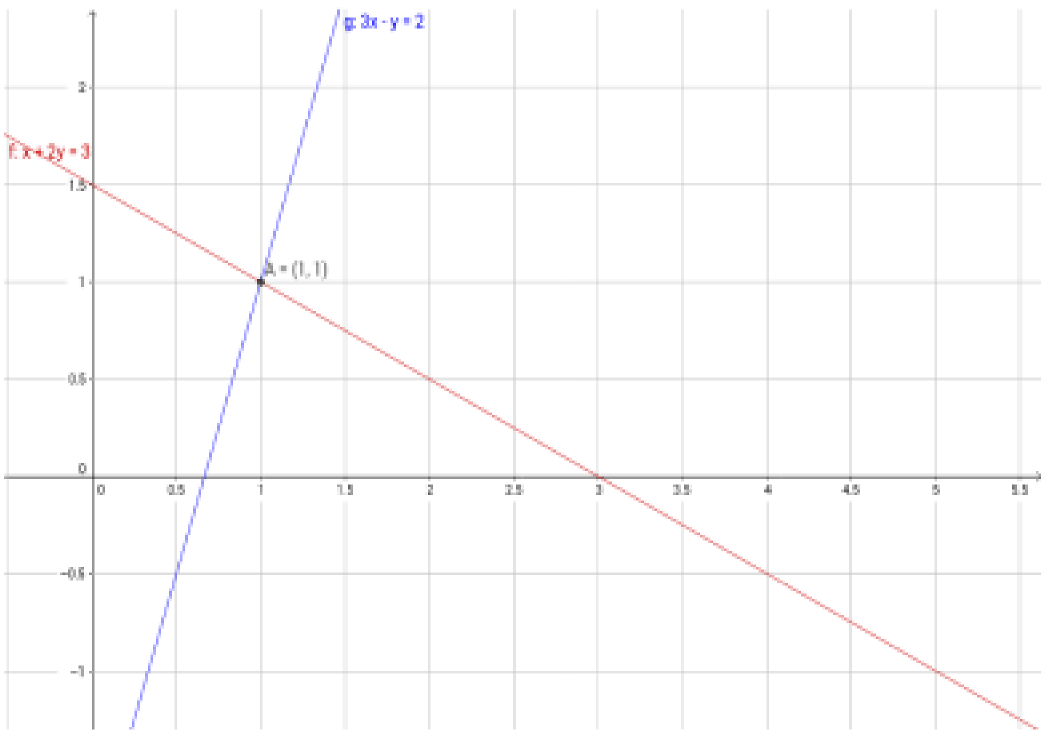
upper triangular system

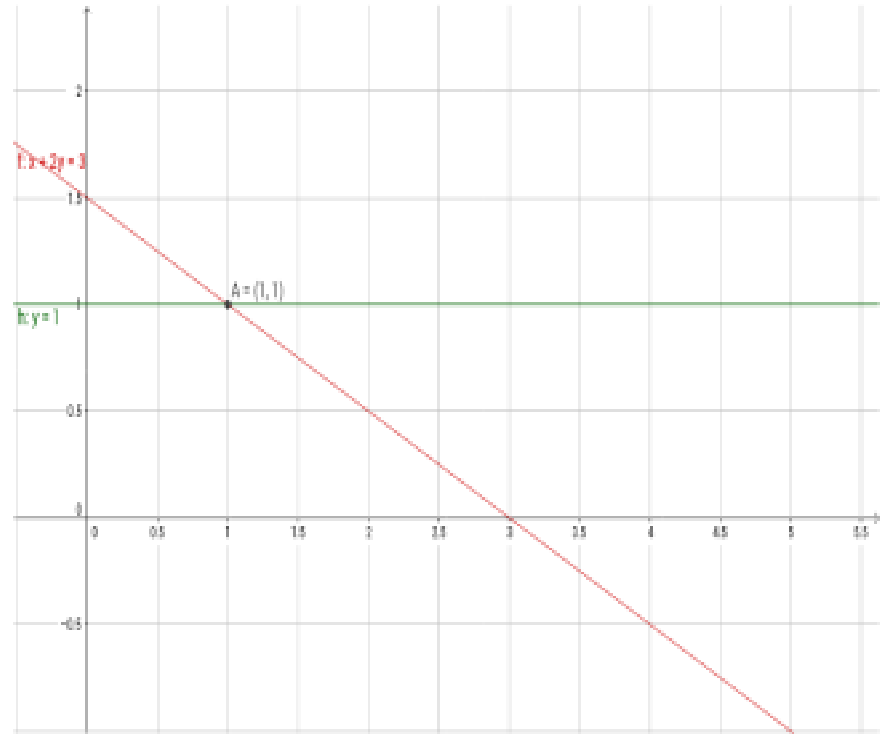
Step 2: "Back-Substitution"

From equation (d) we have $\boxed{y = 1}$. Substitute this in equation (c).

We get $x + 2 = 3$ or $\boxed{x = 1}$.

Geometry of Elimination





Before Elimination

After Elimination

Note: Solution has not changed!

The Terminology of Elimination

pivot

$$\begin{aligned} 2x + 4y &= 6 \\ \boxed{3}x - y &= 2 \end{aligned} \xrightarrow{\text{eliminate } x} \begin{aligned} 2x + 4y &= 6 \\ -7y &= -7 \end{aligned}$$

Subtract multiple of eqn 1 from eqn 2.

$$\text{Multiplier} = \frac{\boxed{3}}{2}$$

pivot

pivots are on diagonal

System is triangular

Pivot = First non zero entry in row that does elimination

Multiplier = (entry to eliminate) divided by (pivot)

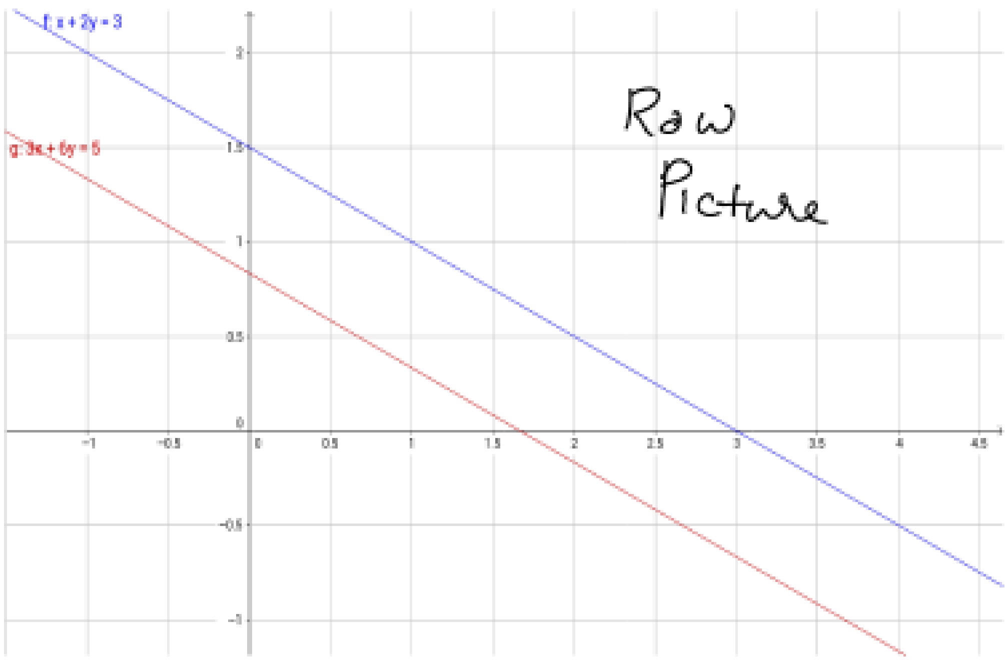
When does Elimination Break Down?

$\boxed{\text{Case 1:}}$ (Permanent Failure) No Solution

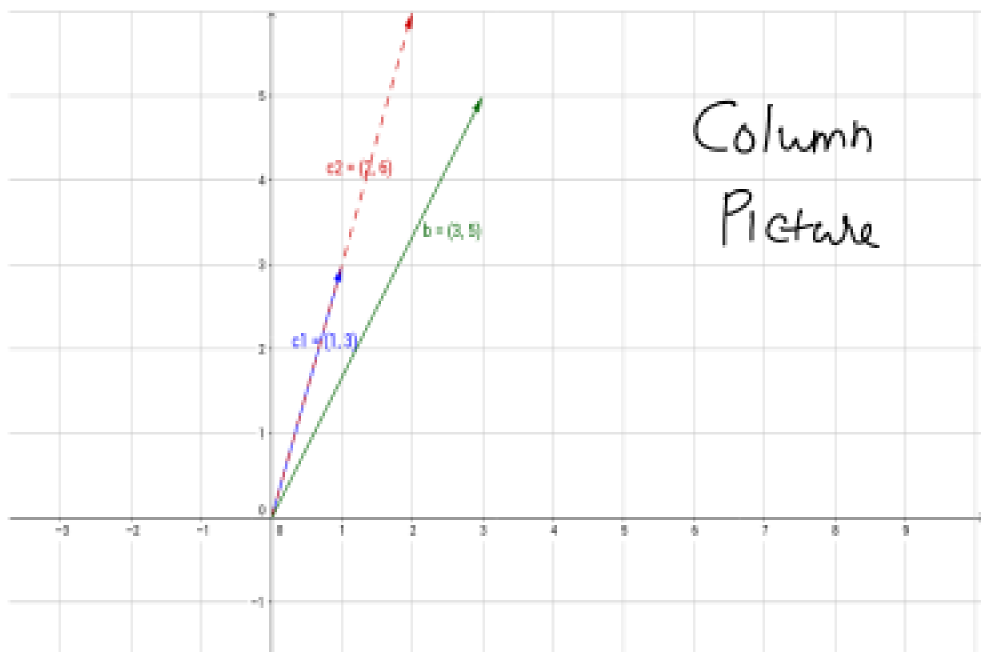
$$\begin{aligned} x + 2y &= 3 \\ 3x + 6y &= 5 \end{aligned} \xrightarrow{\text{elimination}} \begin{aligned} x + 2y &= 3 \\ \boxed{0}\,y &= -4 \end{aligned}$$

zero not allowed as a pivot

LHS of eqn ② = 3 × LHS of eqn ①



Raw Picture



Column Picture

When Does Elimination Break Down?

$\boxed{\text{Case 2:}}$ (Permanent failure) Infinitely many Solutions

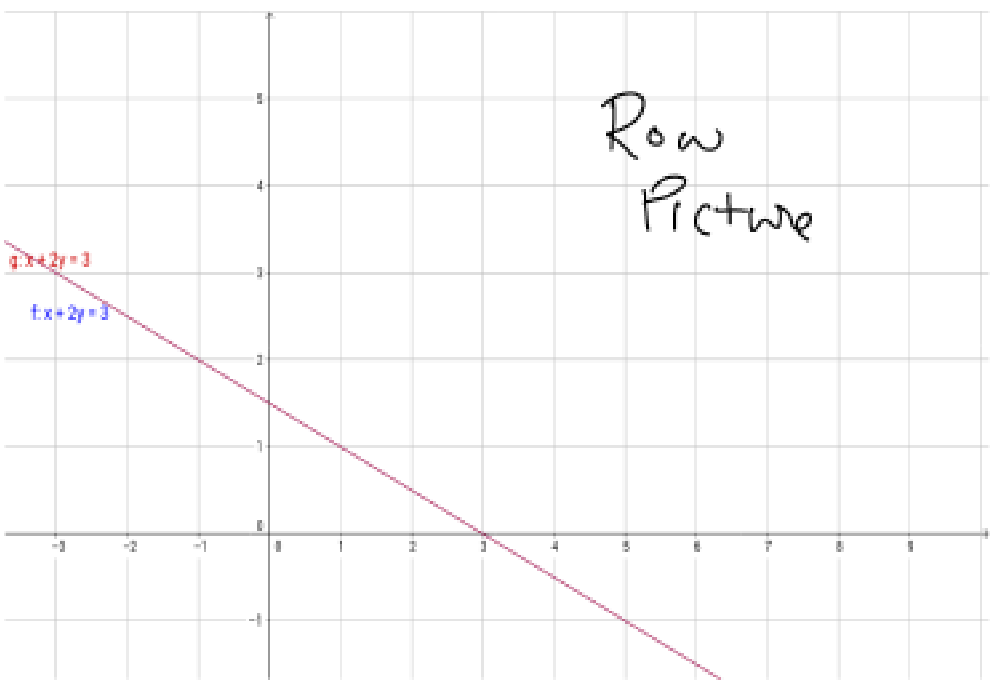
$$\begin{array}{r} x + 2y = 3 \\ 3x + 6y = 9 \end{array} \xrightarrow{\text{elimination}} \begin{array}{r} x + 2y = 3 \\ \boxed{0}y = \boxed{0} \end{array}$$

eqn ② = 3 × eqn ①

every y satisfies $0y = 0$

(y is "free")

Second equation is redundant



In both cases, there was only 1 (non-zero) pivot.

In general, for a system of n equations in n unknowns,

$\underbrace{\text{elimination breaks}}$ down if we do not have n pivots.

no solution or infinitely many solutions } Singular

Note: Temporary failure of Elimination

$$\begin{aligned} \boxed{0}x + 3y &= 3 \\ 3x - y &= 2 \end{aligned} \xrightarrow[\text{rows}]{\text{exchange}} \begin{aligned} 3x - y &= 2 \\ 3y &= 3 \end{aligned}$$

~~zero pivot?~~

non-zero pivots

triangular

non-singular

Elimination works for large systems too!

Example: 3 equations in 3 unknowns

$$
\begin{aligned}
2x + 3y + z &= 4 \\
3x - y - 3z &= -1 \\
2x + 4y + 2z &= 6
\end{aligned}
$$

$\boxed{\text{Step 1a}}$ Eliminate x from the second equation

pivot

$$
\begin{aligned}
\boxed{2}x + 3y + z &= 4 \\
3x - y - 3z &= -1 \\
2x + 4y + 2z &= 6
\end{aligned}
\quad
\xrightarrow[\text{eqn 2} \to \text{eqn 2} - \frac{3}{2}\text{ eqn 1}]{\text{Multiplier } \ell = \frac{3}{2}}
\quad
\begin{aligned}
2x + 3y + z &= 4 \\
-\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\
2x + 4y + 2z &= 6
\end{aligned}
$$

$\boxed{\text{Step 1b}}$ Eliminate x from the third equation

pivot

$$
\begin{aligned}
\boxed{2}x + 3y + z &= 4 \\
-\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\
2x + 4y + 2z &= 6
\end{aligned}
\quad
\xrightarrow[\text{eqn 3} \to \text{eqn 3} - \text{eqn 1}]{\text{Multiplier } \ell = 1}
\quad
\begin{aligned}
2x + 3y + z &= 4 \\
-\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\
y + z &= 2
\end{aligned}
$$

$$\boxed{\begin{aligned} 2x + 3y + z &= 4 \\ 3x - y - 3z &= -1 \\ 2x + 4y + 2z &= 6 \end{aligned}}$$

$$2x + 3y + z = 4$$

$$\left.\begin{aligned} -\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\ y + z &= 2 \end{aligned}\right\} \text{2x2 system}$$

Step 2 $\downarrow$ apply elimination on this system

$\boxed{\text{Step 2}}$ Eliminate y from the third equation above

$$2x + 3y + z = 4$$

$$\boxed{\begin{aligned} \boxed{-\tfrac{11}{2}y} - \tfrac{9}{2}z &= -7 \\ y + z &= 2 \end{aligned}}$$

pivot

$$\xrightarrow[\text{eqn } 3 \,\to\, \text{eqn } 3 - \left(\frac{-2}{11}\right) \text{eqn } 2]{\text{multiplier } \ell = -\frac{2}{11}}$$

$$\begin{aligned} 2x + 3y + z &= 4 \\ -\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\ \tfrac{2}{11}z &= \tfrac{8}{11} \end{aligned}$$

$$\underbrace{\begin{aligned} 2x + 3y + z &= 4 \\ 3x - y - 3z &= -1 \\ x + 4y + 2z &= 6 \end{aligned}}_{A\vec{x} = \vec{b}} \longrightarrow \begin{aligned} 2x + 3y + z &= 4 \\ -\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\ y + z &= 2 \end{aligned} \longrightarrow \begin{aligned} 2x + 3y + z &= 4 \\ -\tfrac{11}{2}y - \tfrac{9}{2}z &= -7 \\ \tfrac{2}{11}z &= \tfrac{8}{11} \end{aligned}$$

upper triangular

$U\vec{x} = \vec{c}$

$\boxed{\text{Step 3}}$ Back Substitution

$$
\begin{aligned}
2x + 3y + z &= 4 \\
-\tfrac{11}{2} y - \tfrac{9}{2} z &= -7 \\
\tfrac{2}{11} z &= \tfrac{8}{11}
\end{aligned}
$$

upper triangular

$U\vec{x} = \vec{c}$

We get (from third equation) $\boxed{z = 4}$

Substituting this in the second equation, we get $-\frac{11}{2} y - 18 = -7$ or $\boxed{y = -2}$

Substituting for z and y in the first equation, we get $2x - 6 + 4 = 4$ or $\boxed{x = 3}$

Worked Examples (#7 in Problem Set 2.2)

For which numbers a does elimination break down (1) permanently (2) temporarily?

$$ax + 3y = -3$$
$$4x + 6y = 6.$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

Worked Examples (#7 in Problem Set 2.2)

For which numbers a does elimination break down (1) permanently (2) temporarily?

$$ax + 3y = -3$$
$$4x + 6y = 6.$$

Solve for x and y after fixing the temporary breakdown by a row exchange.

Observe LHS of the two equations.

What value of a makes LHS of eqn ② a multiple of eqn ①?

$\boxed{a=2}$

If $a = 2$, then

LHS eqn ② $= 2 \times$ LHS eqn ①

Permanent Breakdown

$$\begin{array}{l} 2x + 3y = -3 \\ 4x + 6y = 6 \end{array} \xrightarrow{\text{elimination}} \begin{array}{l} 2x + 3y = -3 \\ 0x + 0y = 12 \\ \hline 0 = 12 \end{array}$$

NO SOLUTION

breakdown

$$ax + 3y = -3$$
$$4x + 6y = 6.$$

Temporary breakdown of elimination

- should be fixable by a row exchange

Elimination breaks down temporarily if $\boxed{a=0}$.

$$\begin{aligned} 0x + 3y &= -3 \\ 4x + 6y &= 6 \end{aligned} \xrightarrow[\text{exchange}]{\text{row}} \begin{aligned} 4x + 6y &= 6 \\ 3y &= -3 \end{aligned} \xrightarrow[\text{Substitution}]{\text{Back}} \boxed{\begin{aligned} y &= -1 \\ x &= 3 \end{aligned}}$$

triangular

Worked Examples (#13 in Problem Set 2.2)

Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ______ times row ______ from row ______.

Worked Examples (#13 in Problem Set 2.2)

Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \quad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract _______ times row _______ from row _______.

$$\begin{aligned} 2x - 3y \quad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned} \quad \xrightarrow[\text{from row 2}]{\text{Subtract } 2 \times \text{row 1}} \quad \begin{aligned} 2x - 3y \quad &= 3 \\ y + z &= 1 \\ 2x - y - 3z &= 5 \end{aligned} \quad \xrightarrow[\text{from row 3}]{\text{Subtract } 1 \times \text{row 1}} \quad \begin{aligned} 2x - 3y \quad &= 3 \\ y + z &= 1 \\ 2y - 3z &= 2 \end{aligned}$$

Worked Examples (#13 in Problem Set 2.2)

Apply elimination (circle the pivots) and back substitution to solve

$$\begin{aligned} 2x - 3y \quad &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5. \end{aligned}$$

List the three row operations: Subtract ______ times row ______ from row ______ .

$$\begin{aligned} 2x - 3y \quad &= 3 \\ y + z &= 1 \\ 2y - 3z &= 2 \end{aligned} \xrightarrow[\text{from row 3}]{\text{Subtract } 2 \times \text{row 2}} \begin{aligned} \boxed{2}x - 3y \quad &= 3 \\ \boxed{1}y + z &= 1 \\ \boxed{-5}z &= 0 \end{aligned} \xrightarrow[\text{substitution}]{\text{Back}} \boxed{\begin{aligned} z &= 0 \\ y &= 1 \\ x &= 3 \end{aligned}}$$

pivots