# LECTURE 4

9/9/16

## LAST TIME

- Matrices
- Inverse
- Independence, Depedence

## TODAY

Linear Equations (#2.1 in text)

# Linear Equations

An example: $y = -x + 7$

$y = 2x + 4$

What is the solution?

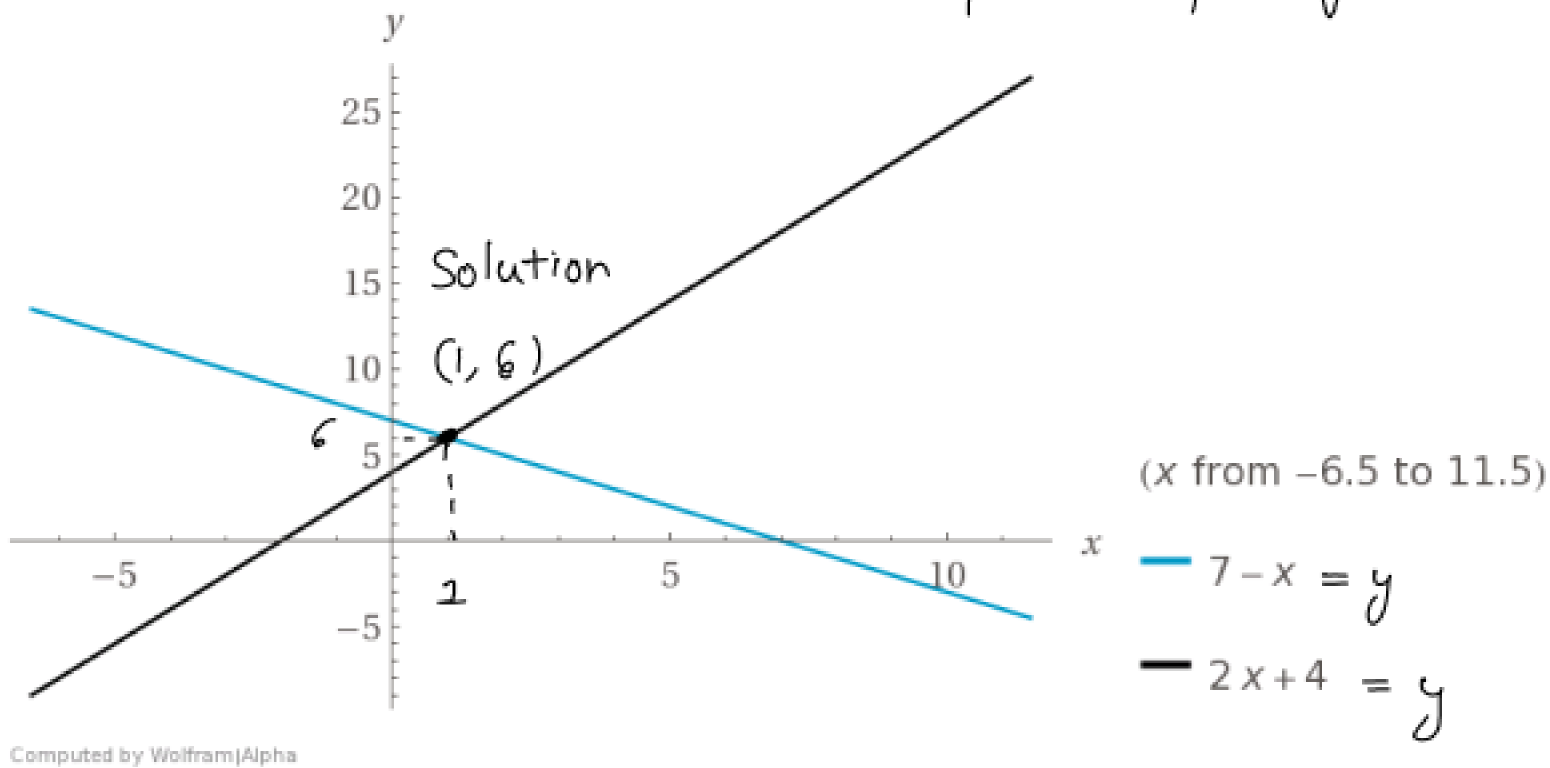
$x = 1$ and $y = 6$

Why is this a linear equation?

- coefficients of $x, y$ are scalars (real numbers)
- the equations do not have terms of the form $x^2$ or $xy$

# The Row Picture

Consider each row of the system of equations




(1, 6) lies on both lines

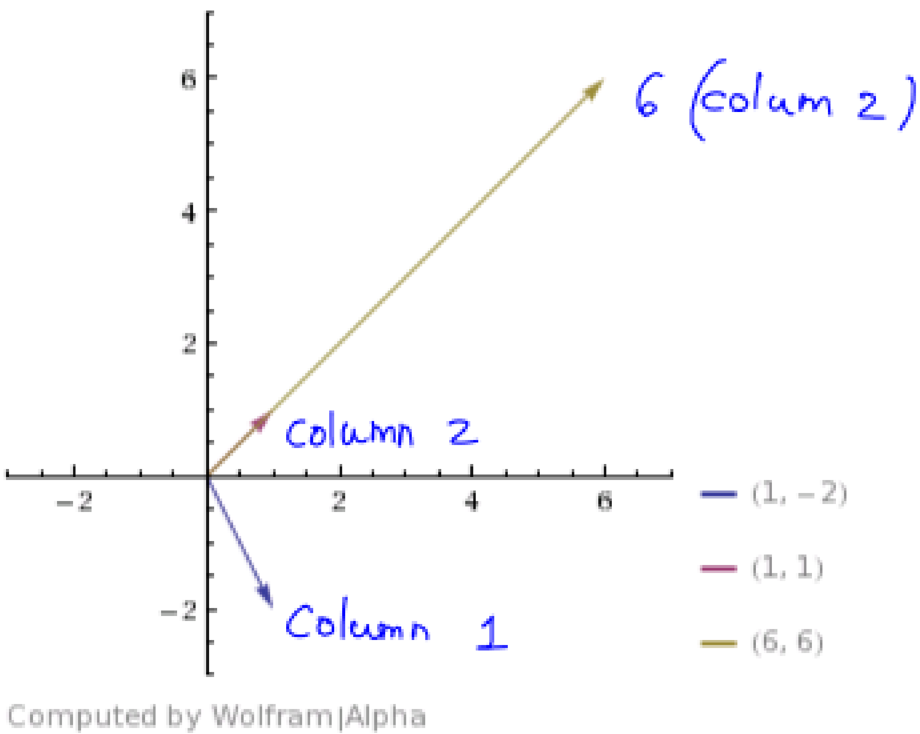
<u>Question:</u> Is there always a solution? What if the lines are parallel?

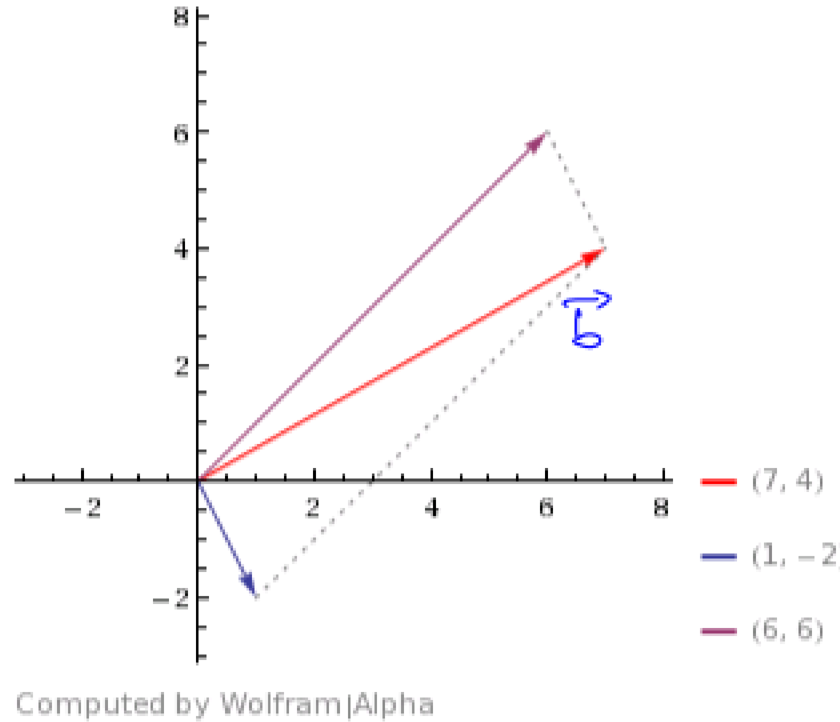
## The Column Picture

Rewrite the system of equations as a linear combination of vectors

$$x\begin{bmatrix}1\\-2\end{bmatrix} + y\begin{bmatrix}1\\1\end{bmatrix} = \underbrace{\begin{bmatrix}7\\4\end{bmatrix}}_{\vec{b}}$$

Find the correct linear combination (i.e., values of $x$ and $y$) which yields $\vec{b}$




1 (column 1) + 6 (colum 2) = $\vec{b}$

## Some Notation

Matrix Equation

$$\begin{aligned} y &= -x + 7 \\ y &= 2x + 4 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}}_{\substack{\text{Coefficient} \\ \text{matrix, } A}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

## Interpretation as a function

$$\underset{\vec{x}}{\begin{bmatrix}1\\6\end{bmatrix}} \xrightarrow{A = \begin{bmatrix}1 & 1\\-2 & 1\end{bmatrix}} \underset{\vec{b}}{\begin{bmatrix}7\\4\end{bmatrix}}$$

to get $\vec{b}$ as the output from A, we must input $\vec{x}$

<u>Note:</u> All three viewpoints give the same solution $\vec{x}$

# Some Geometry ...

In the two-dimensional plane (in $\mathbb{R}^2$), inner products give lines

Why? from our example

$$\underbrace{(\text{row } 1)}_{\substack{\text{of} \\ \text{coefficient} \\ \text{matrix}}} \cdot \vec{x} = (1, 1) \cdot (x, y) = \underbrace{x + y = b_1}_{\substack{\text{equation of} \\ \text{a line}}}$$

How about in three-dimensional space (in $\mathbb{R}^3$)?

# Three Equations in Three Unknowns

**Example:**

$$x + 2y + 3z = -3$$
$$3x + 2y + z = 3$$
$$2x + y + 3z = -3$$

Matrix equation $A\vec{x} = \vec{b}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}}_{\substack{\text{coefficient} \\ \text{matrix} \\ A}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}}_{\vec{b}}$$

# The Row Picture

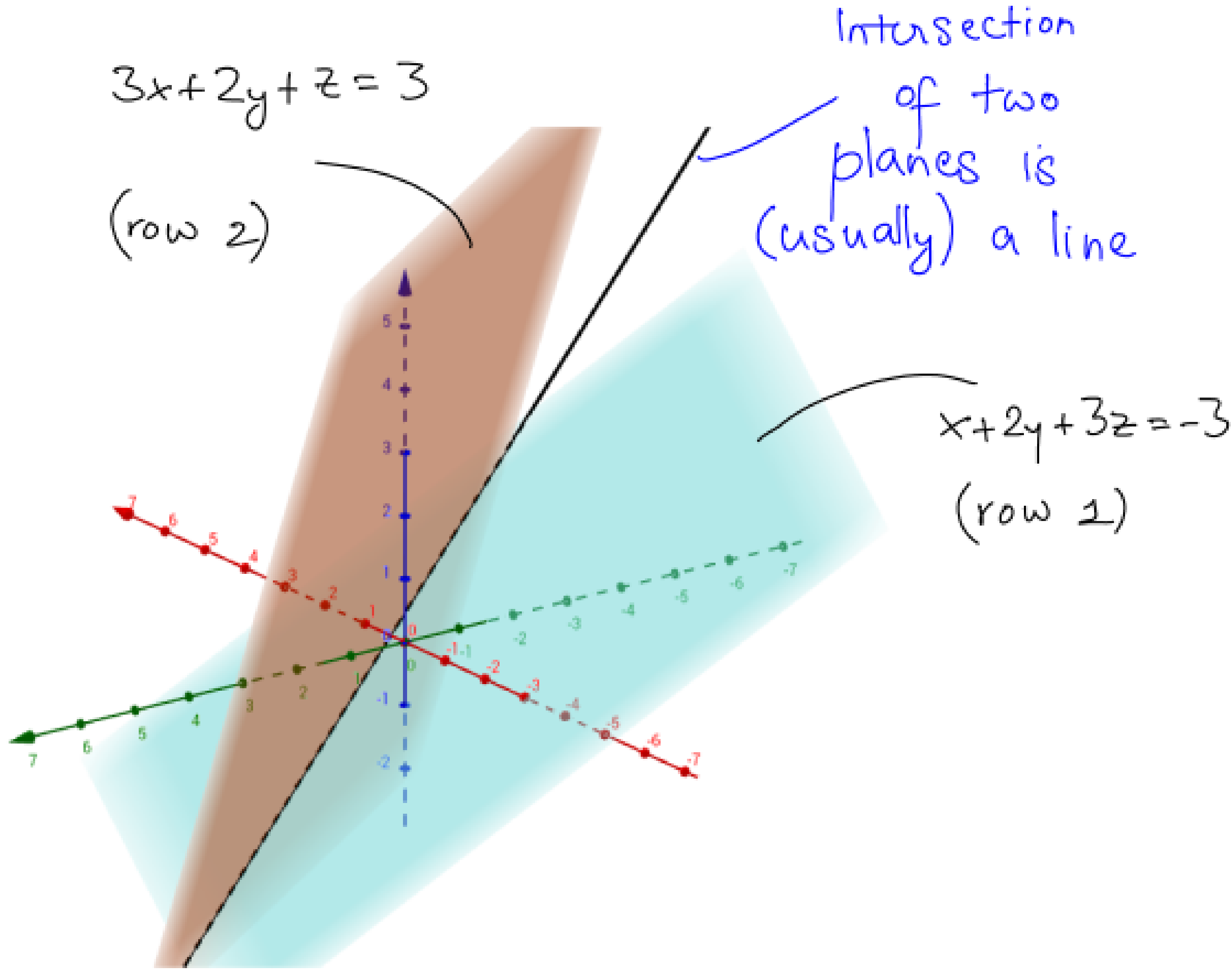
Consider the first row of the system of equations

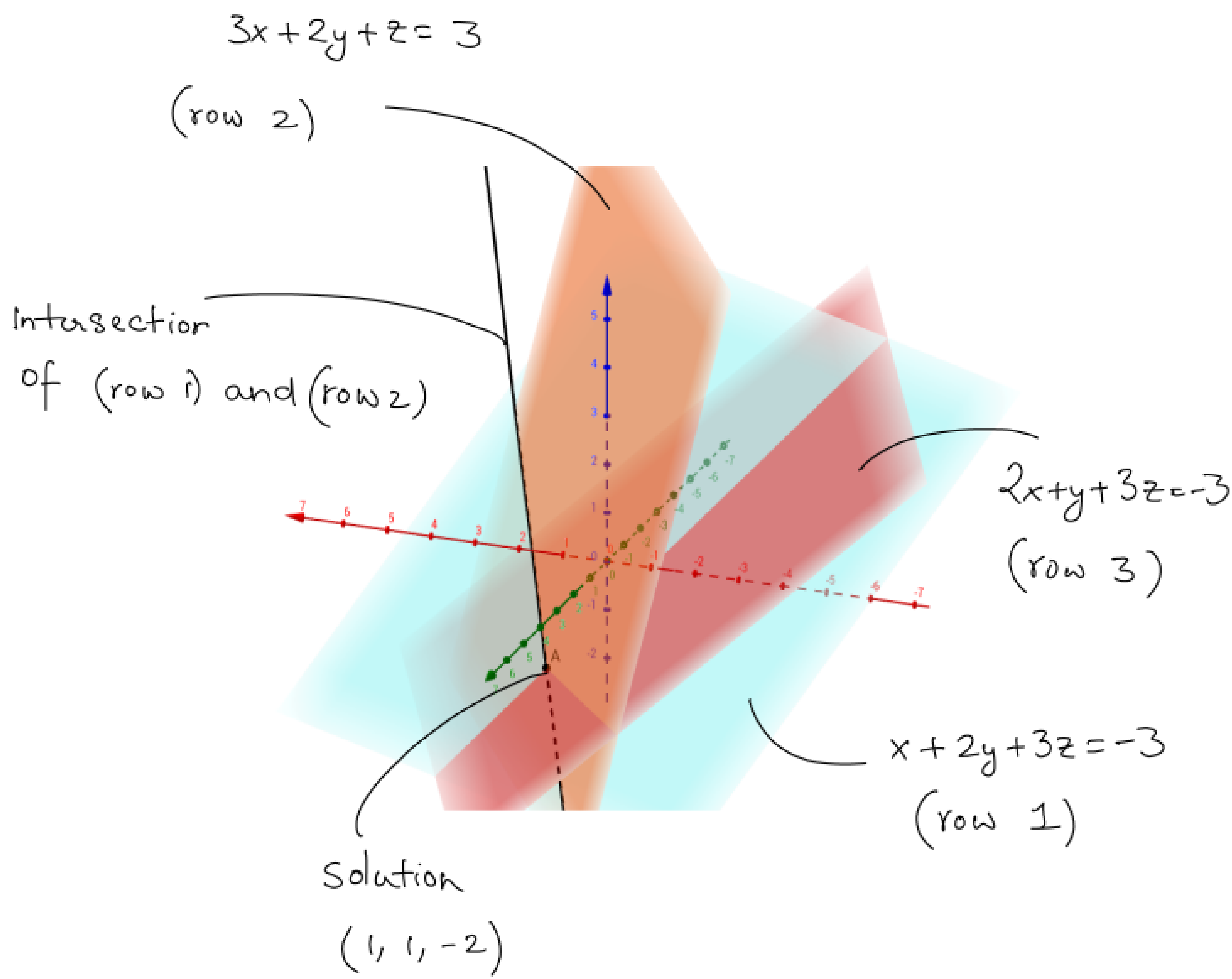
$$x + 2y + 3z = -3,$$

or $$\underbrace{(1, 2, 3)}_{\text{row } 1} \cdot \underbrace{(x, y, z)}_{\vec{x}} = -3$$

this describes a plane in three-dimensional space

i.e., Inner products in $\mathbb{R}^3$ $$\underbrace{\begin{pmatrix} a \\ b \\ c \end{pmatrix}}_{\text{coefficients}} \cdot \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{variables}} = d$$ gives planes.


$3x+2y+z=3$
(row 2)
Intersection of two planes is (usually) a line
$x+2y+3z=-3$
(row 1)

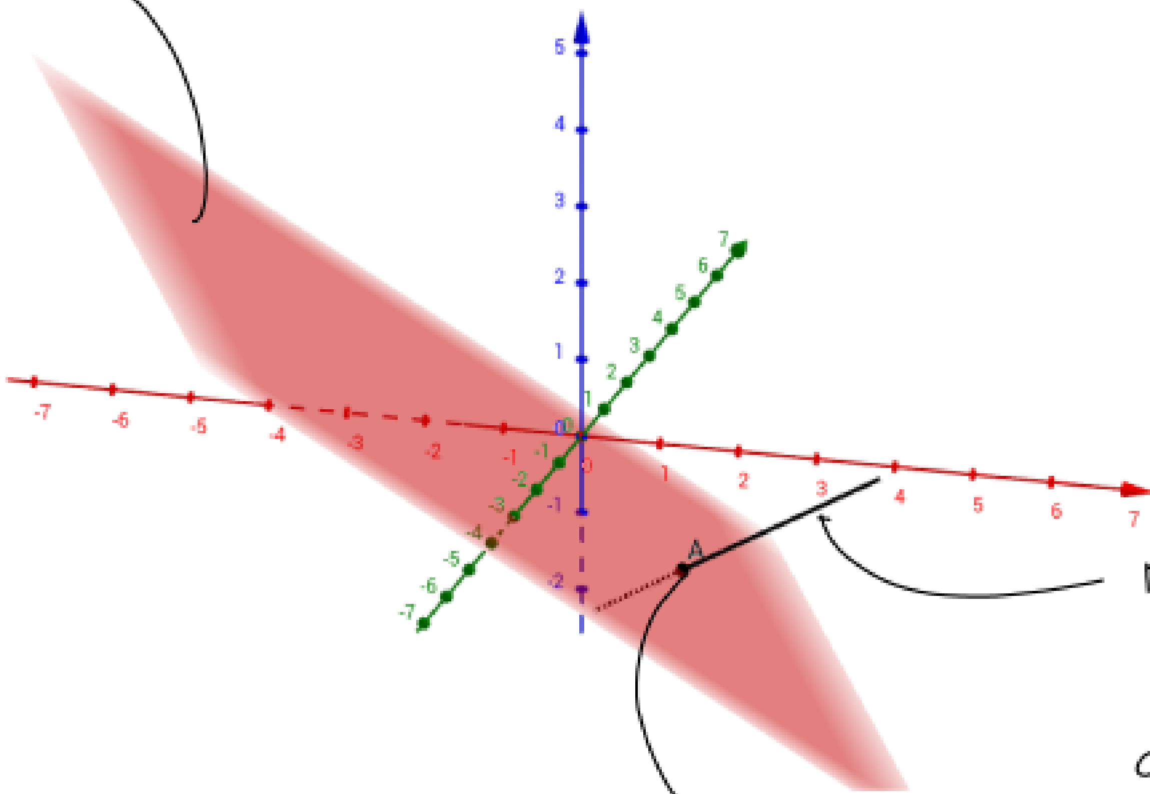

$3x+2y+z=3$
(row 2)
Intersection of (row 1) and (row 2)
$2x+y+3z=-3$
(row 3)
$x+2y+3z=-3$
(row 1)
A
Solution
$(1, 1, -2)$

**Question:**

Is there always a solution?

$2x + y + 3z = -3$

(row 3)



Intersection of (row 1) and (row 2)

Solution

(1, 1, −2)

## Column Picture

$$\begin{aligned} x + 2y + 3z &= -3 \\ 3x + 2y + z &= 3 \\ 2x + y + 3z &= -3 \end{aligned}$$

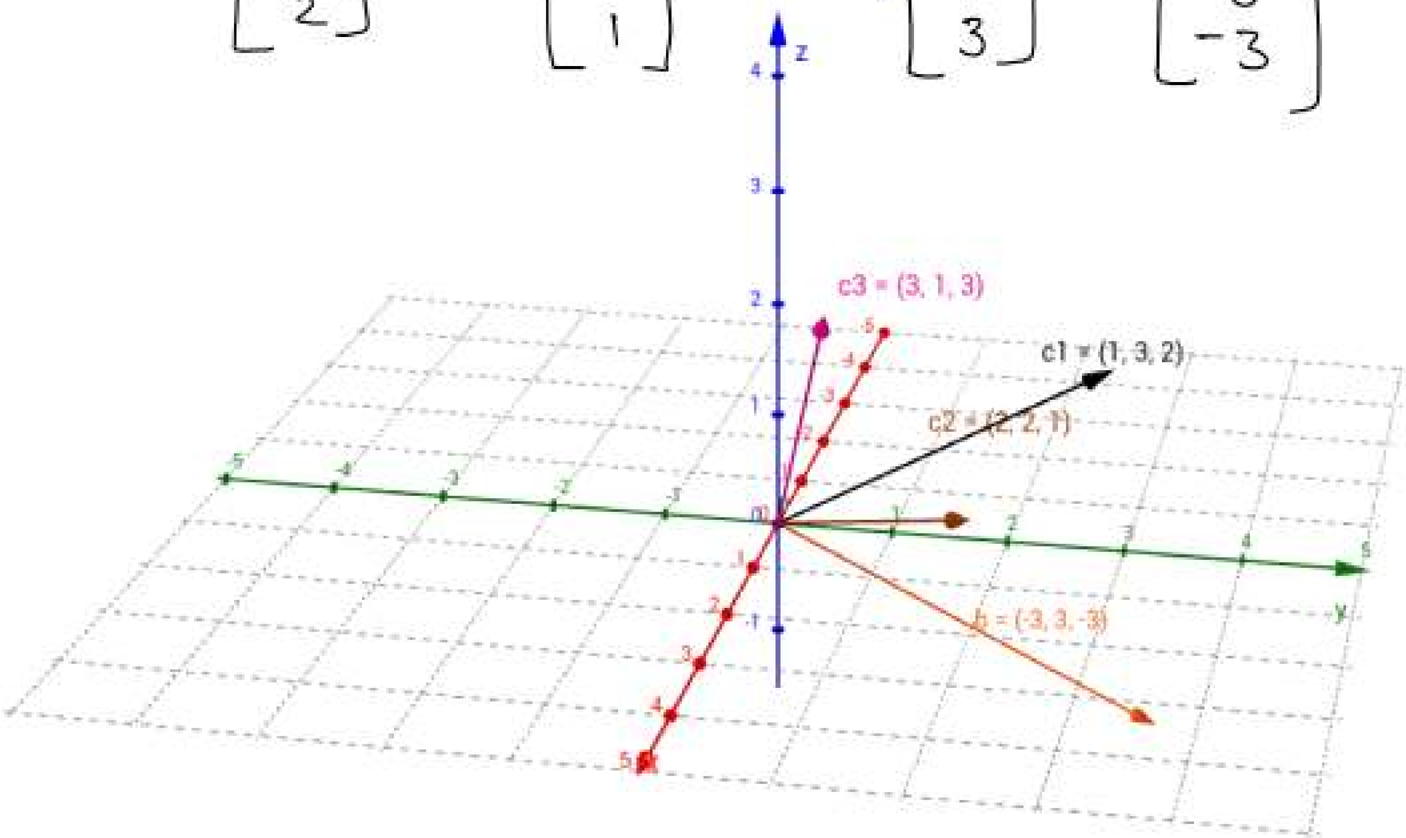
Vector form of the equations $A\vec{x} = \vec{b}$

$$x \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + z \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}$$

Which linear combination (i.e., choices of $x, y, z$) produce $\vec{b}$?

Correct combination is

$$\underline{1}\begin{bmatrix}1\\3\\2\end{bmatrix} + \underline{1}\begin{bmatrix}2\\2\\1\end{bmatrix} + (-2)\begin{bmatrix}3\\1\\3\end{bmatrix} = \begin{bmatrix}-3\\3\\-3\end{bmatrix}$$



An Example where there is no solution...

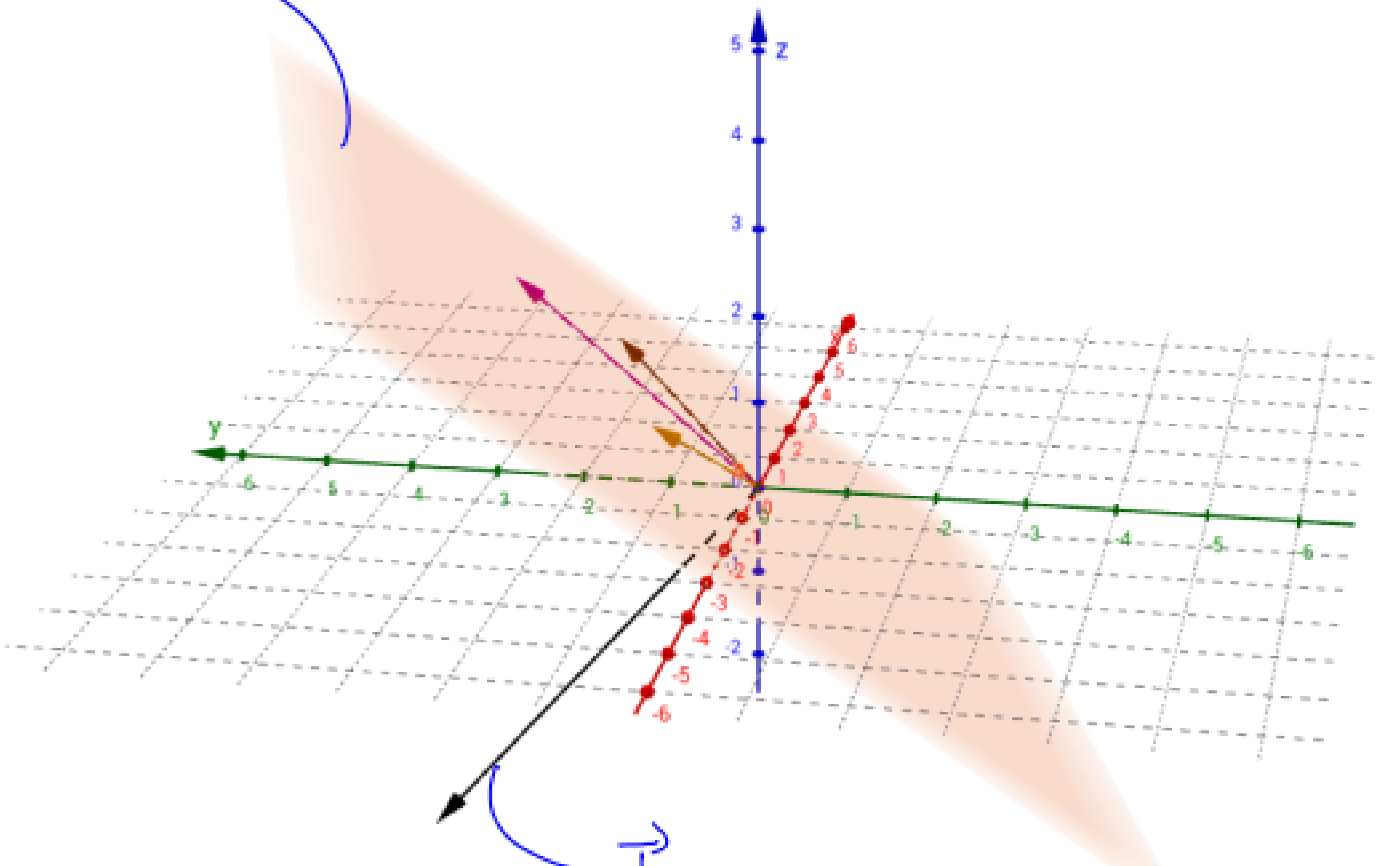
$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}}_{\vec{b}}$$

col 3 = col 1 − col 2

all columns of the coefficient matrix A lie on a plane, but $\vec{b}$ does not lie on this plane

# the column picture

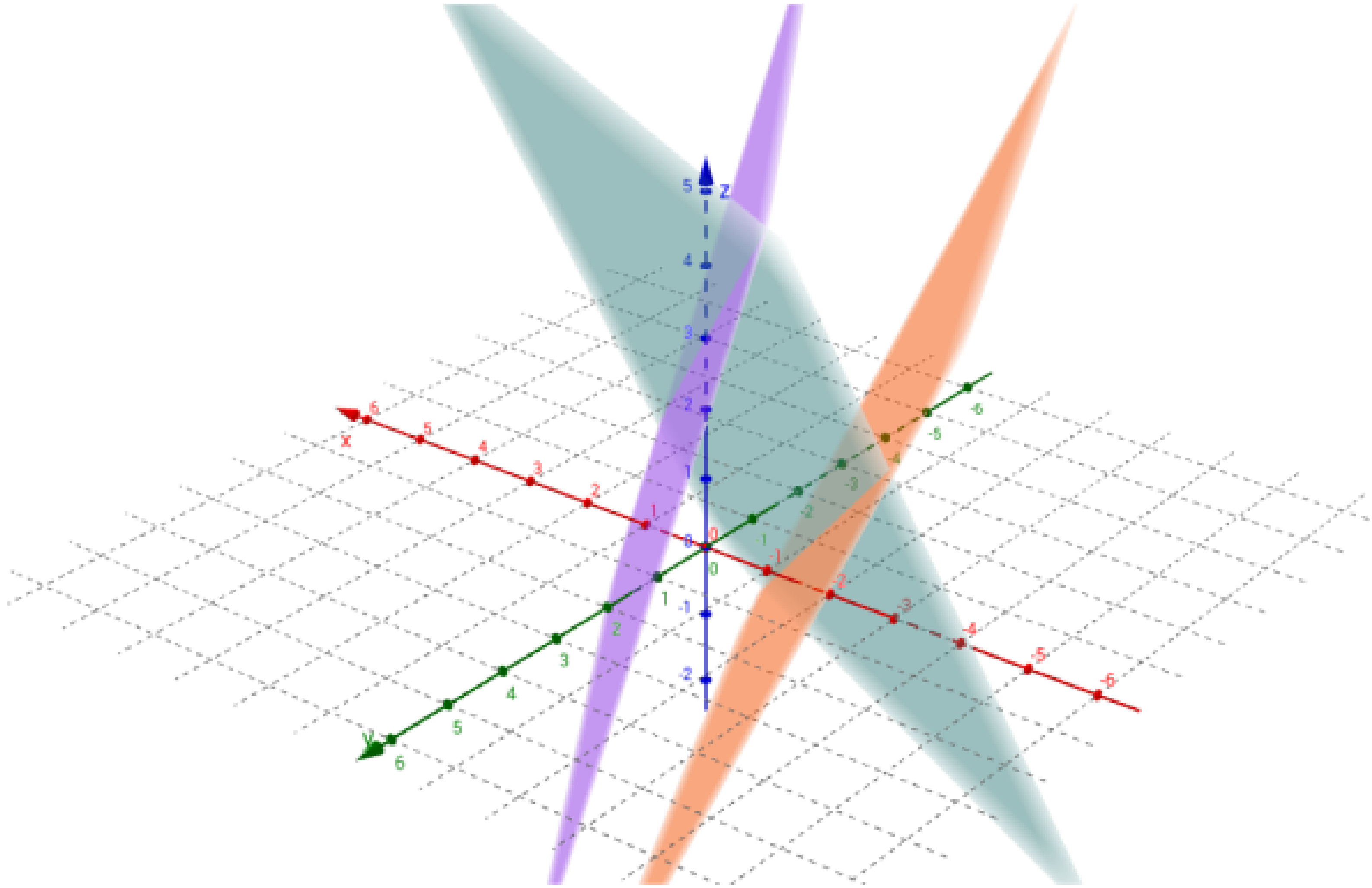
plane containing col 1, col 2, col 3




$\vec{b}$

(does not lie in plane)

the corresponding <u>row picture</u>



all the planes do not intersect at a single point

An example where there are infinitely many solutions.

$$\underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\vec{b}}$$

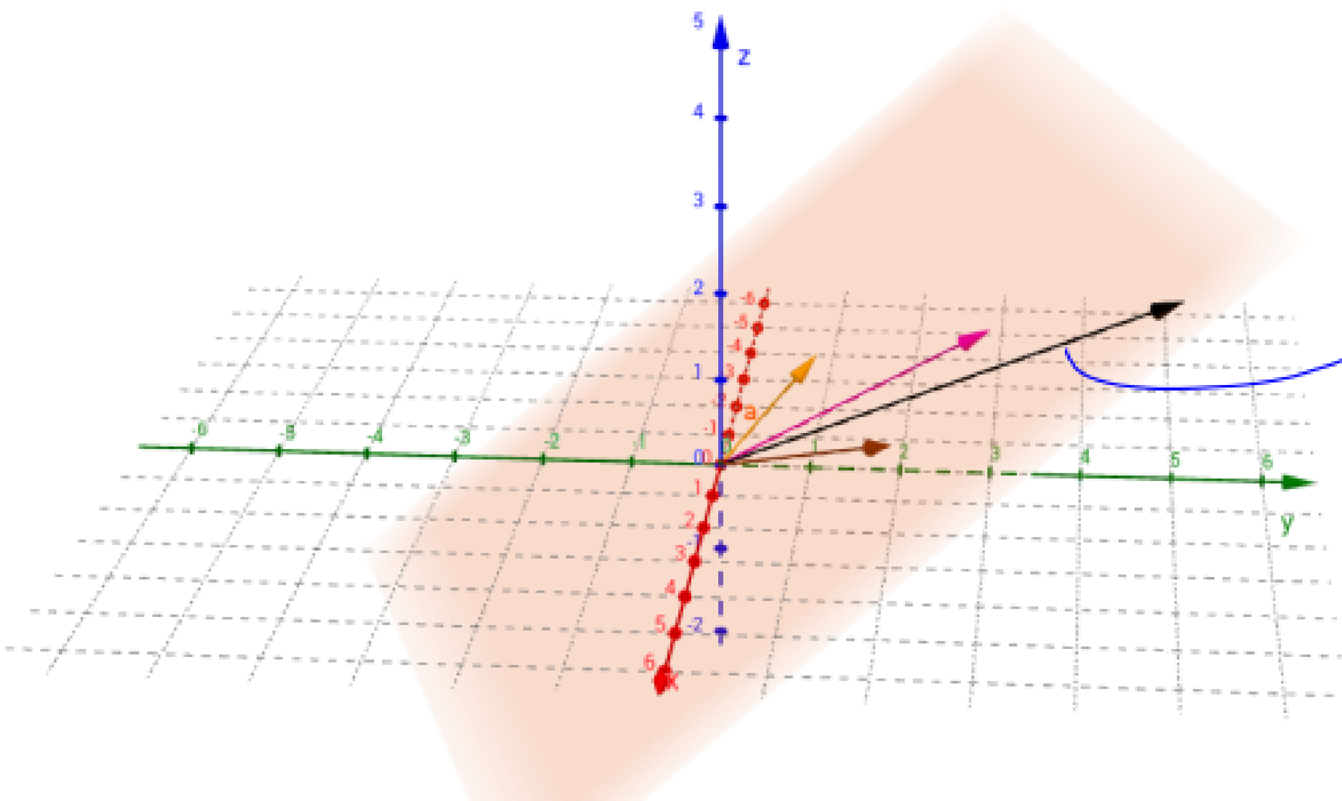
col 3 = col 1 − col 2

$\vec{b}$ is now a linear combination of only col 1 and col 2

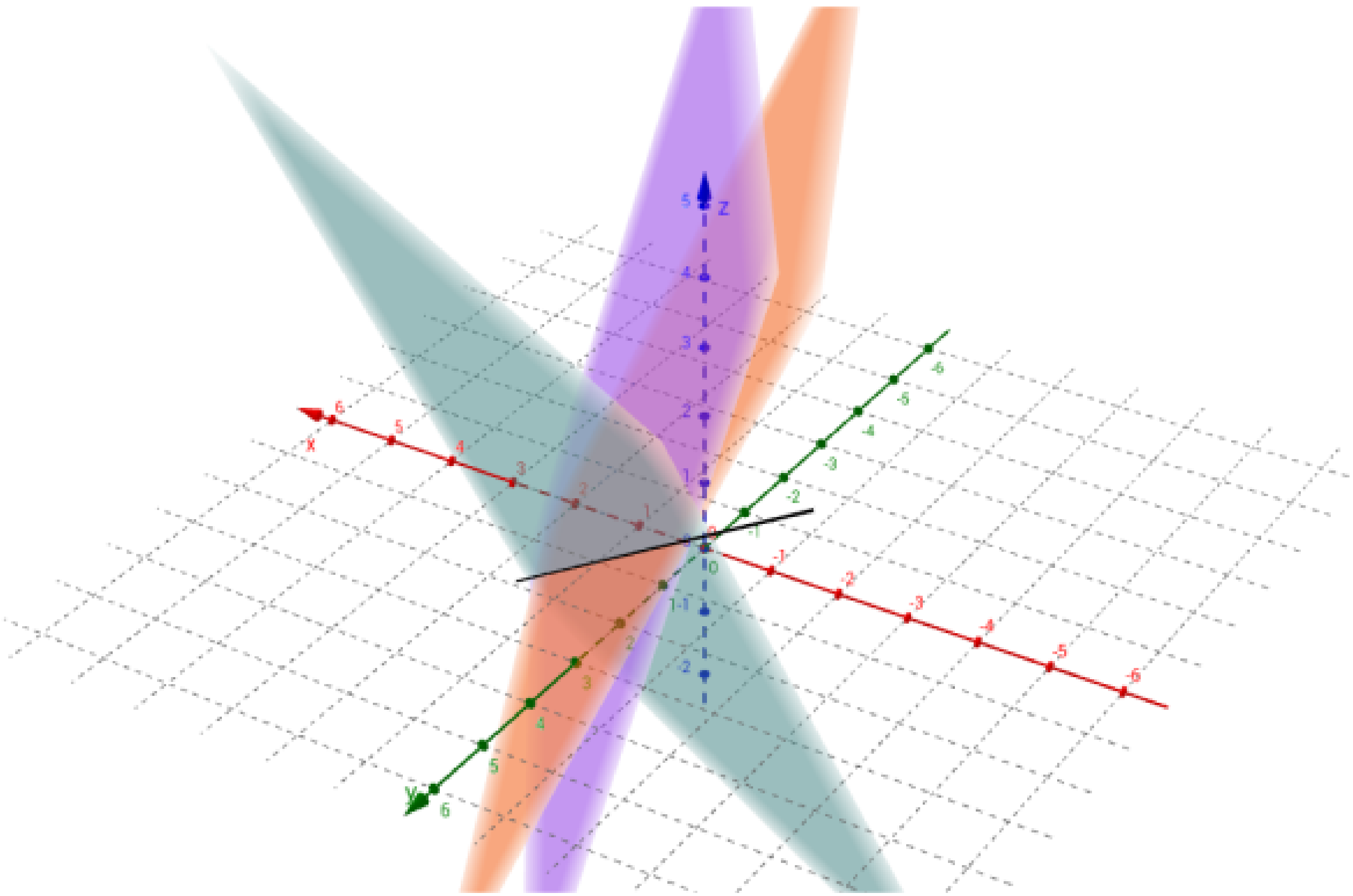
all columns of the coefficient matrix A lie on a plane, and $\vec{b}$ also lies on this plane

$$\underbrace{1\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + 0\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\text{Linear combination 1}} \qquad \underbrace{2\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}}_{\text{Linear combination 2}}$$

Infinitely many solutions!




$\vec{b}$ is now in the plane which contains col 1, col 2 and col 3 the corresponding <u>row picture</u>




all three planes intersect along a line!

# Function Interpretation

$$\underset{\text{(input)}}{\overset{?}{\vec{x}}} \qquad A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \qquad \underset{\text{(output)}}{\underset{\vec{b}}{\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix}}}$$

what input gives the output $\vec{b}$ ?

# The Identity Matrix

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

main diagonal of the matrix

<u>Property</u> $I\vec{x} = \vec{x}$ for all vector $\vec{x} \in \mathbb{R}^3$.

- the vector is not changed by multiplication with $I$
- like multiplication with 1, but for matrices and vectors.

# Matrix Notation

Consider a 2 by 2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Entry in $i^{th}$ row and $j^{th}$ column denoted as $a_{ij}$

column number

row number

Sometimes, we write $A(i,j)$ instead of $a_{ij}$

Worked Example (#7 from Problem Set 2.1)

Consider the linear system of equations

$$x + y + z = 2$$
$$x + 2y + z = 3$$
$$2x + 3y + 2z = 5$$

The columns of the coefficient matrix are (1,1,2) and (1,2,3) and (1,1,2)

This is a $\boxed{\text{singular}}$ case because the third column is __________

Find two combinations of the columns that give $\vec{b} = (2,3,5)$. This is only possible for $\vec{b} = (4, 6, c)$ if $c =$ __________

## Worked Example (#7 from Problem Set 2.1)

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The columns of the coefficient matrix are $(1,1,2)$ and $(1,2,3)$ and $(1,1,2)$

This is a $\boxed{\text{singular}}$ case because the third column is = col 1

Find two combinations of the columns that give $\vec{b} = (2,3,5)$.

This is only possible for $\vec{b} = (4,6,c)$ if $c =$ 10

two possible combinations

$$1\begin{pmatrix}1\\1\\2\end{pmatrix} + 1\begin{pmatrix}1\\2\\3\end{pmatrix} + 0\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}$$

$$0\begin{pmatrix}1\\1\\2\end{pmatrix} + 1\begin{pmatrix}1\\2\\3\end{pmatrix} + 1\begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}$$

If $c=10$, $\vec{b} = (4,6,c)$ lies in the plane of the columns

## Worked Example (#33, Problem Set 2.1)

Let $A$ be some $2\times 2$ matrix. If $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $A\vec{u}$ and $A\vec{v}$ are the columns of $A$. Combine $\vec{w} = c\vec{u} + d\vec{v}$. If $\vec{w} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$, how is $A\vec{w}$ connected to $A\vec{u}$ and $A\vec{v}$?

## Worked Example (#33, Problem Set 2.1)

Let $A$ be some $2\times 2$ matrix. If $\vec{u}=\begin{bmatrix}1\\0\end{bmatrix}$ and $\vec{v}=\begin{bmatrix}0\\1\end{bmatrix}$ then $A\vec{u}$ and $A\vec{v}$ are the columns of $A$. Combine $\vec{w}=c\vec{u}+d\vec{v}$. If $\vec{w}=\begin{bmatrix}5\\7\end{bmatrix}$, how is $A\vec{w}$ connected to $A\vec{u}$ and $A\vec{v}$?

We see that $\vec{w}=5\begin{bmatrix}1\\0\end{bmatrix}+7\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}5\\7\end{bmatrix}$.

Let $A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$. Then $A\vec{w}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}5\\7\end{bmatrix}=\begin{bmatrix}5a_{11}+7a_{12}\\ 5a_{21}+7a_{22}\end{bmatrix}$

$$A\vec{u}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}a_{11}\\ a_{21}\end{bmatrix}$$

$$A\vec{v}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}a_{12}\\ a_{22}\end{bmatrix}$$

We have

$$A\vec{w}=5\begin{bmatrix}a_{11}\\ a_{21}\end{bmatrix}+7\begin{bmatrix}a_{12}\\ a_{22}\end{bmatrix}$$

$$\underline{A\vec{w}=5\,A\vec{u}+7\,A\vec{v}}$$

# Linearity

Multiplying by $A$ is a $\boxed{\text{linear transformation}}$

$\boxed{\text{If } \vec{w} \text{ is a linear combination of } \vec{u} \text{ and } \vec{v}\text{, then } A\vec{w} \text{ is the same linear combination of } A\vec{u} \text{ and } A\vec{v}}$