LECTURE 3

9/7/2016

LAST TIME

- Dot product
- Length / Norm
- Angle between vectors
- Cauchy-Schwartz and Triangle Inequalities

Today

- Matrices
- Inverse of a matrix
- Dependence and Independence

(Recall) Linear Combinations

<u>Example</u>

$$3\begin{pmatrix}1\\1\\1\end{pmatrix}+(-1)\begin{pmatrix}2\\1\\0\end{pmatrix}+2\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}3\\0\\7\end{pmatrix}$$

We can represent this linear combination as a matrix-vector system

$$\begin{bmatrix}1&2&1\\1&1&-1\\1&0&2\end{bmatrix}\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}3\\0\\7\end{bmatrix}$$

In general, in n-dimensional space,

linear combinations of vectors look like

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \ldots\ldots + c_m \vec{v}_m = \sum_{j=1}^{m} c_j \vec{v}_j, \quad \text{where}$$

- $c_1, c_2, \ldots\ldots, c_m$ are scalars (real numbers)
- $\vec{v}_1, \vec{v}_2, \ldots\ldots, \vec{v}_m$ are vectors in n-dimensional space ($\mathbb{R}^n$)

In matrix-vector notation, this linear combination can be written as

$$\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

<u>Note:</u> matrix has n rows and m columns.

Interpretations of the Matrix-Vector System

(I) as a linear combination of the columns

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + x_2 \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where $\vec{v} = (v_1, v_2)$

$\vec{w} = (w_1, w_2)$

$\vec{x} = (x_1, x_2)$

② As the dot product with rows

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (v_1, w_1) \cdot (x_1, x_2) \\ (v_2, w_2) \cdot (x_1, x_2) \end{bmatrix}$$

$$= \begin{bmatrix} \text{row } \underline{1} \cdot \vec{x} \\ \text{row } 2 \cdot \vec{x} \end{bmatrix}$$

(II) **As a function**

$$\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} v_1 x_1 + w_1 x_2 \\ v_2 x_1 + w_2 x_2 \end{bmatrix}$$

Function — Input — Output

Denote $M = \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$. We can interpret the matrix-vector system as a function $M : \mathbb{R}^2 \to \mathbb{R}^2$

We write $M \in \mathbb{R}^{2\times 2}$ (matrices with 2 rows and 2 columns)

Inputs are ordered pairs of real numbers (vectors in $\mathbb{R}^2$)

Outputs are also ordered pairs of real numbers (vectors in $\mathbb{R}^2$)

$$\underbrace{\begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} \underbrace{v_1 x_1 + w_1 x_2}_{b_1} \\ \underbrace{v_2 x_1 + w_2 x_2}_{b_2} \end{bmatrix} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}}_{\vec{b}}$$

or $\quad M\vec{x} = \vec{b}$

Two types of problems:

① M and $\vec{x}$ are known; find $\vec{b}$

the "forward" problem

② M and $\vec{b}$ are known; find $\vec{x}$

the "inverse" problem

Example. Can you "design" a matrix M which performs the following transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow[?]{M} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

Inputs Outputs

(Switch the 1st and 3rd entries)

Here is such a matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix}$$

Inverse of a Matrix

From our previous example,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \xrightarrow{M} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} \xrightarrow{?} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(1^{st} and 3^{rd} components switched)

Can we undo the switching?

In this case, Yes!

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{M} \left(\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} \right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$

We will say that a matrix $A \in \mathbb{R}^{n \times n}$ is <u>invertible</u>

if there exists another matrix B such that

$$B(A\vec{x}) = \vec{x} \quad \text{for all } \vec{x} \in \mathbb{R}^n.$$

<u>Notation:</u> The inverse of A is denoted as A^{-1}.

* In our previous example, the switch matrix was its own inverse.

* <u>Note:</u> Not every matrix is invertible!

Consider the matrix which "transforms"

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \xrightarrow{E} \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

(i.e., erase the 1^{st} entry)

Here is such a matrix

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix}$$

This matrix is not invertible

Why is E not invertible?

Consider the equation $E\vec{x} = \vec{b}$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

or

$$\begin{pmatrix} 0x_1 + 0x_2 + 0x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Consider the "inverse" problem: Given $\vec{b}$, we want to find $\vec{x}$

<u>Case 1</u> If $b_1 \neq 0$, then there is <u>no solution</u>

<u>Case 2</u> If $b_1 = 0$, then there are <u>infinitely many solutions</u>

Independence and Dependence

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

M is inventible

Columns of M are

$$\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\vec{u}_1$ is not in the plane $\vec{u}_2$ and $\vec{u}_3$

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E is not inventible

Columns of E are

$$\vec{a}_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{a}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$\vec{a}_1$ is in the plane of $\vec{a}_2$ and $\vec{a}_3$

$(\vec{a}_1 = 0\,\vec{a}_2 + 0\,\vec{a}_3)$

Independence and Dependence

$$M = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

M is invertible

Columns of M are

<u>independent</u>

no combination except

$$0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$$

$$E = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

E is not invertible

Columns of E are

<u>dependent</u>

other combinations give $\vec{0}$

for example

$$2\vec{a}_1 + 0\vec{a}_2 + 0\vec{a}_3 = \vec{0}$$

Example (#4 from Problem Set 1.3)

Find a combination $x_1 \vec{w_1} + x_2 \vec{w_2} + x_3 \vec{w_3}$ that gives the zero vector.

$$\vec{w_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{w_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \vec{w_3} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

Those vectors are (independent) (dependent).

The three vectors lie in a __________

The matrix W with those columns is __________

Example (#4 from Problem Set 1.3)

Find a combination $x_1 \vec{w_1} + x_2 \vec{w_2} + x_3 \vec{w_3}$ that gives the zero vector.

$$\vec{w_1} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \vec{w_2} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \qquad \vec{w_3} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

Those vectors are ~~(independent)~~ (dependent).

The three vectors lie in a plane

The matrix W with those columns is not invertible

Clearly, $0\vec{w_1} + 0\vec{w_2} + 0\vec{w_3} = \vec{0}$. But are there other combinations?

we see that $\vec{w_2} = \frac{1}{2}\vec{w_1} + \frac{1}{2}\vec{w_3}$.

Therefore $-\frac{1}{2}\vec{w_1} + \vec{w_2} - \frac{1}{2}\vec{w_3} = \vec{0}$