$\boxed{\text{LECTURE 2}}$

9/1/2016

LAST TIME

* Vectors (in $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$)
* Vector addition, scalar multiplication
* Linear combinations

Today

* Dot Product
* Length or Norm
* Unit vectors

Review Problem (#6 in Problem Set 1.1)

Every combination of $\vec{v} = (1, -2, 1)$ and $\vec{w} = (0, 1, -1)$ has components that add to ______

Find c and d so that $c\vec{v} + d\vec{w} = (3, 3, -6)$.

Soln: $c\vec{v} + d\vec{w} = c\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c \\ -2c+d \\ c-d \end{pmatrix}$

Hence, components add up to

$$c + (-2c+d) + (c-d) = \boxed{0}$$

If $c\vec{v} + d\vec{w} = (3, 3, -6)$, then

$c = 3$ and $c - d = -6$, or $d = c + 6 = 3 + 6 = 9$

$\boxed{c = 3}$ and $\boxed{d = 9}$

The Dot Product

$\boxed{\text{Def}^n}$ If $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$ are two vectors, their dot product is denoted $\vec{a} \cdot \vec{b}$ and is defined by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k .$$

Note: * For $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$,

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

* $\vec{a} \cdot \vec{b}$ is a scalar!

Some Properties of the Dot Product

(Here, $\vec{a}$, $\vec{b}$, $\vec{c}$ are vectors and η is a scalar)

(i) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ (commutative law)

(ii) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ (distributive law)

(iii) $\eta\,(\vec{a} \cdot \vec{b}) = (\eta \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\eta \vec{b})$

(iv) $\vec{a} \cdot \vec{a} > 0$ if $\vec{a} \neq \vec{0}$

(v) $\vec{a} \cdot \vec{a} = 0$ if $\vec{a} = \vec{0}$

Examples:

① If $\vec{a} = (3, 2)$ and $\vec{b} = (4, -6)$, then

$$\vec{a} \cdot \vec{b} = (3)(4) + (2)(-6)$$

$$= 12 + (-12)$$

$$= 0.$$

Length or Norm

Defⁿ If $\vec{a}$ is a vector, its length or norm is denoted by $\|\vec{a}\|$ and is defined by

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}}$$

Note: for a vector $\vec{a} \in \mathbb{R}^2$ $\qquad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$

$\vec{a} = (a_1, a_2)$ (two-dimensions)

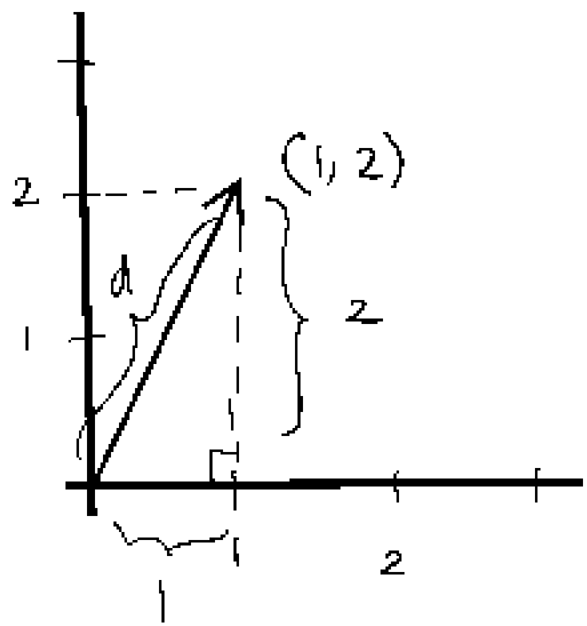
for a vector $\vec{a} \in \mathbb{R}^3$ $\qquad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\vec{a} = (a_1, a_2, a_3)$ (three-dimensions)

for a vector $\vec{a} \in \mathbb{R}^n$ $\qquad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$\vec{a} = (a_1, a_2, \ldots\ldots, a_n)$ (n-dimensions) this should not be surprising

for $\vec{a} = (1, 2)$



By the Pythogorean theorem,

$$d = \sqrt{1^2 + 2^2} = \sqrt{5}$$

similar argument holds in three dimensions: apply pythogorean theorem twice

Some properties

If $\vec{a}$ is a vector and c is a scalar,

we have the following properties:

(a) $\|\vec{a}\| > 0$ if $\vec{a} \neq \vec{0}$ (positivity)

(b) $\|\vec{a}\| = 0$ if $\vec{a} = \vec{0}$

(c) $\|c\vec{a}\| = |c|\,\|\vec{a}\|$ (homogeneity)

Unit Vectors

Def^n A unit vector $\vec{u}$ is a vector whose length equals 1.

i.e. $\vec{u} \cdot \vec{u} = 1$

For example, in three-dimensional space,

$$\vec{u} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

is a unit vector.

Why? $\vec{u} \cdot \vec{u} = \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = 1$

Hence $\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}} = 1.$

Some Important Unit Vectors

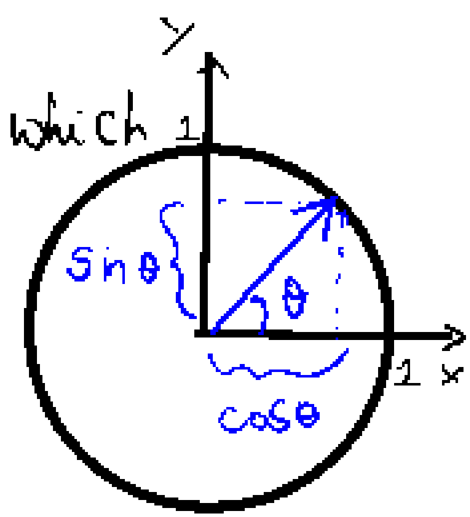
In the two-dimensional plane, any vector $\vec{v} = (v_1, v_2)$ can be written as

$$(v_1, v_2) = v_1(1,0) + v_2(0,1)$$

The two vectors $(1,0)$ and $(0,1)$ are unit coordinate vectors along the x and y axes.

$$i = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Similarly, $(\cos\theta, \sin\theta)$ is a unit vector which makes an angle θ with the x axis



In three-dimensional space, the standard unit vectors are

$$i = (1,0,0) \qquad j = (0,1,0) \qquad k = (0,0,1)$$

In general, in n-dimensions,

$$\vec{e}_1 = (1,0,0,\ldots\ldots,0)$$
$$\vec{e}_2 = (0,1,0,\ldots\ldots,0)$$
$$\vdots$$
$$\vec{e}_k = (0,0,\ldots,0,\underset{k^{th}\text{ component}}{1},0,\ldots,0)$$
$$\vdots$$
$$\vec{e}_n = (0,0,\ldots\ldots,0,1)$$

note: $\|\vec{e}_k\| = 1$ and

$$\vec{e}_k \cdot \vec{e}_j = 0 \text{ if } k \neq j$$

Every vector $\vec{v} = (v_1, v_2, \ldots, v_n)$ can be expressed in the form

$$\vec{v} = v_1 \vec{e}_1 + v_2 \vec{e}_2 + \ldots + v_n \vec{e}_n = \sum_{k=1}^{n} v_k \vec{e}_k .$$

Obtaining a unit vector along $\vec{v}$

Given a vector $\vec{v}$

$$\boxed{\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} \quad \text{is a unit vector in the same direction as } \vec{v}.}$$

i.e., divide any non-zero vector $\vec{v}$ by its length $\|\vec{v}\|$

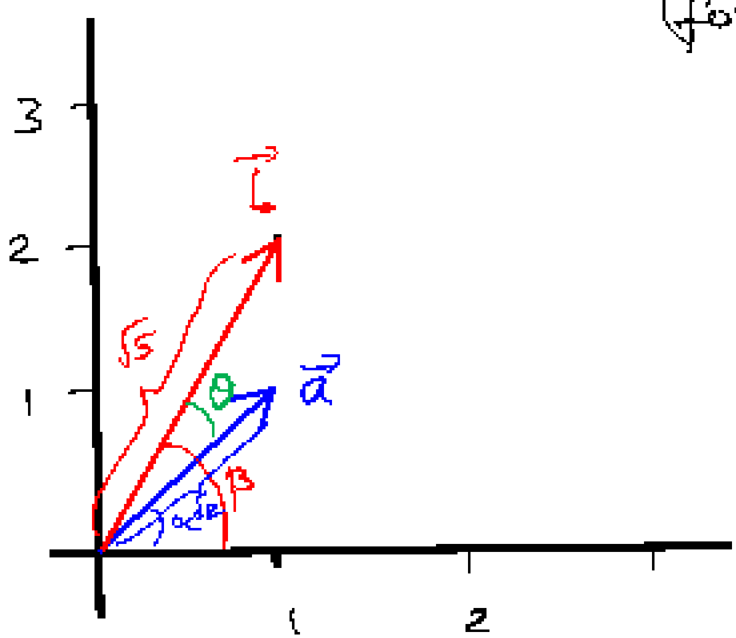
Example: Let $\vec{a} = (1, 2, 2)$.

then $\|\vec{a}\| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{1+4+4} = \sqrt{9} = 3$.

the vector $\vec{u} = \dfrac{\vec{a}}{\|\vec{a}\|} = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$ or $\dfrac{(1,2,2)}{3}$ is a unit vector along the same direction as $\vec{a}$.

Angle between Vectors

Consider $\vec{a} = (1,1)$ and $\vec{b} = (1,2)$



Recall: $\cos\theta = \dfrac{\text{adjacent side length}}{\text{hypotenuse length}}$

(for a right triangle) $\sin\theta = \dfrac{\text{opposite side length}}{\text{hypotenuse length}}$

$\cos\alpha = \dfrac{1}{\sqrt{2}} = \dfrac{1}{\|\vec{a}\|}$ $\quad \cos\beta = \dfrac{1}{\sqrt{5}} = \dfrac{1}{\|\vec{b}\|}$

$\sin\alpha = \dfrac{1}{\sqrt{2}}$ $\quad \sin\beta = \dfrac{2}{\sqrt{5}}$

$$\cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha$$

$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{5}}\frac{1}{\sqrt{2}}$$

$$= \frac{(1)(1) + (2)(1)}{\sqrt{5}\sqrt{2}} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

For any two non-zero vectors $\vec{a}, \vec{b}$, the angle between $\vec{a}, \vec{b}$ satisfies

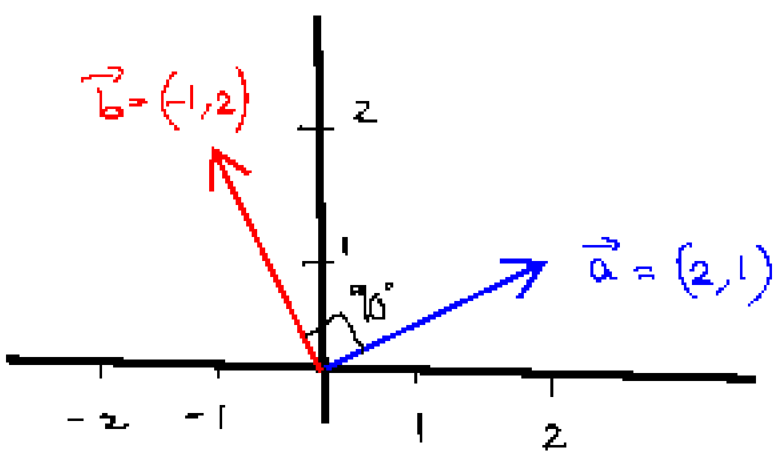
$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$$

Note: another way of writing this: $\vec{a}\cdot\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\cos\theta$

What happens when $\theta = 90°$?

When $\vec{a}$ is perpendicular to $\vec{b}$, the dot product $\vec{a}\cdot\vec{b} = 0$

For example



$\vec{a} = (2, 1)$ and $\vec{b} = (-1, 2)$

$\vec{a}\cdot\vec{b} = (2)(-1) + (1)(2) = 0$

$$\boxed{\text{Cauchy - Schwartz Inequality}}$$

Recall: angle between two non-zero vectors $\vec{a}$ and $\vec{b}$

satisfies $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|}$.

Since $|\cos\theta| \leq 1$,

$$\boxed{|\vec{a}\cdot\vec{b}| \leq \|\vec{a}\|\,\|\vec{b}\|}$$

Alternate expressions. $(\vec{a}\cdot\vec{b})^2 \leq (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{b})$

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

Triangle Inequality

If $\vec{a}$ and $\vec{b}$ are vectors, we have

$$\boxed{\|\vec{a}+\vec{b}\| \leq \|\vec{a}\| + \|\vec{b}\|}$$

Proof sketch: To avoid square roots let's work with $\|\vec{a}+\vec{b}\|^2 \leq (\|\vec{a}\| + \|\vec{b}\|)^2$

$$LHS = \|\vec{a}+\vec{b}\|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = \vec{a}\cdot\vec{a} + \underbrace{2\,\vec{a}\cdot\vec{b}}_{=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}} + \vec{b}\cdot\vec{b}$$

using def^n of length

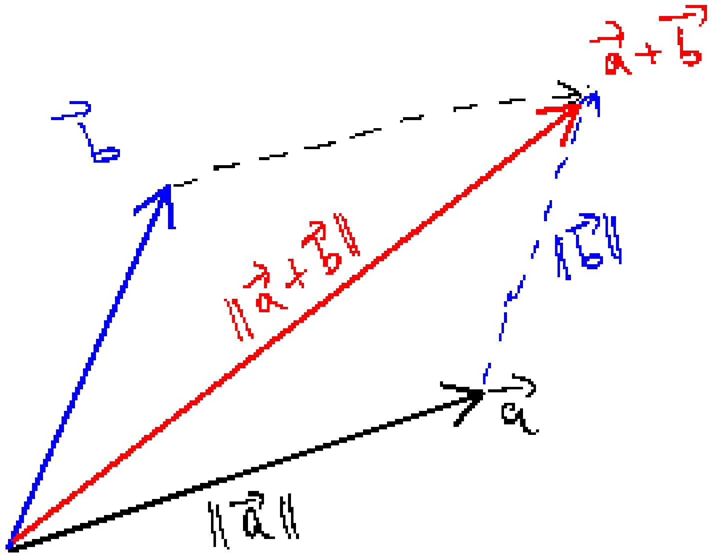
(commutativity)

$$= \|\vec{a}\|^2 + 2\,\vec{a}\cdot\vec{b} + \|\vec{b}\|^2$$

$$RHS = \|\vec{a}\|^2 + 2\|\vec{a}\|\|\vec{b}\| + \|\vec{b}\|^2$$

LHS $\leq$ RHS if $\vec{a}\cdot\vec{b} \leq \|\vec{a}\|\|\vec{b}\|$ apply Cauchy-Schwartz

Geometric Interpretation



Worked Examples

(#5 from Problem Set 1.2)

Find unit vectors $\vec{u}_1$ and $\vec{u}_2$ in the directions of $\vec{v} = (1, 3)$ and $\vec{w} = (2, 1, 2)$. Find unit vectors $\vec{U}_1$ and $\vec{U}_2$ that are perpendicular to $\vec{u}_1$ and $\vec{u}_2$.

Solution: $\vec{u}_1 = \dfrac{\vec{v}}{\|\vec{v}\|}$. Since $\|\vec{v}\| = \sqrt{1^2+3^2} = \sqrt{10}$, $\underline{\vec{u}_1 = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)}$.

$\vec{u}_2 = \dfrac{\vec{w}}{\|\vec{w}\|}$. Since $\|\vec{w}\| = \sqrt{2^2+1^2+2^2} = 3$, $\underline{\vec{u}_2 = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)}$.

Note that $\vec{q} = (-3, 1)$ is perpendicular to $\vec{v}$ since $\vec{q} \cdot \vec{v} = (-3)(1) + (3)(1) = 0$.

We can choose $\underline{\vec{U}_1} = \dfrac{\vec{q}}{\|\vec{q}\|} = \underline{\left(\frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)}$ since $\|\vec{q}\| = \sqrt{(-3)^2 + 1^2} = \sqrt{10}$.

Similarly, $\vec{r} = (1, 0, -1)$ is perpendicular to $\vec{w}$ since $\vec{r} \cdot \vec{w} = (2)(1) + (1)(0) + (2)(-1) = 0$.

We can choose $\underline{\vec{U}_2} = \dfrac{\vec{r}}{\|\vec{r}\|} = \underline{\left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)}$ since $\|\vec{r}\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$.