Topics: - Vectors
- Linear Combinations   #1.1 and #1.2 of text
- Dot product and length

Vectors

Two-dimensional vectors (or vectors in $\mathbb{R}^2$)

a two-dimensional vector can be written as

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

v_1 - first component
v_2 - second component

typically written in boldface
overhead arrow for pen/paper writing

note: written as a column vector

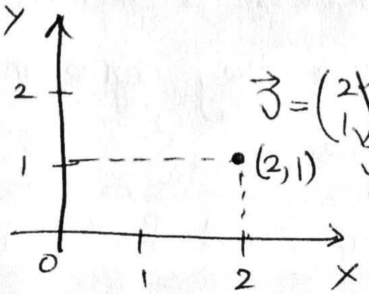
Alternate notation: $\vec{v} = (v_1, v_2)$ - an ordered pair Note: not the same as a row vector

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

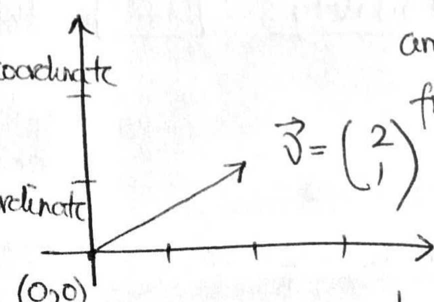
Interpretations of a vector

$\vec{v}$ can be interpreted as
(i) a point the two-dimensional plane
(ii) an arrow in the plane
(iii) a list of two numbers.

Example: $\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$\vec{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ x coordinate, y coordinate



an "arrow" from (0,0) to (2,1)

* this arrow has a length and direction

Vectors in Three Dimensions

- xy plane replaced by xyz space (or in $\mathbb{R}^3$)

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ or $\vec{v} = \underbrace{(v_1, v_2, v_3)}_{\text{"ordered triple"}}$ corresponds to a point in three dimensional space.

$\vec{v}$ can be represented by an arrow from the "origin" with coordinates $(0,0,0)$ to the point with coordinates (v_1, v_2, v_3)

x coordinate, y coordinate, z coordinate

In general, a vector in "n-dimensions" $\vec{v} \in \mathbb{R}^n$ is the vector with components $v_1, v_2, \ldots, v_n$

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ or $\vec{v} = \underbrace{(v_1, v_2, \ldots\ldots, v_n)}_{\text{"ordered n-tuple"}}$

Operations on Vectors

Set Equality $\boxed{\text{Def}^n}$ Two vectors $\vec{a}$ and $\vec{b}$ are called equal whenever they agree in their respective components.

That is, if $\vec{a} = (a_1, a_2, \ldots, a_n)$ and $\vec{b} = (b_1, b_2, \ldots, b_n)$, the vector equation $\vec{a}=\vec{b}$ means exactly the same as the n scalar equations

$$a_1=b_1, \quad a_2=b_2, \; \ldots\ldots, \; a_n=b_n.$$

Vector Addition $\boxed{\text{Defn}}$ The sum of two vectors $\vec{a}$ and $\vec{b}$ is the vector obtained by adding the corresponding components.

$$\vec{a}+\vec{b} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{pmatrix}.$$

Scalar Multiplication $\boxed{\text{Defn}}$ If c is a scalar, we define $c\vec{a}$ to be the vector ~~defined~~ obtained by multiplying each component of $\vec{a}$ by c.

$$c\vec{a} = (ca_1, ca_2, \ldots\ldots, ca_n)$$

Note: The word "scalar" is used here as a synonym for "real number".

From these definitions, it is easy to verify the following properties:

(i) vector addition is commutative: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$

(ii) ... and associative $\vec{a}+(\vec{b}+\vec{c}) = (\vec{a}+\vec{b})+\vec{c}$

(iii) multiplication by scalar is associative $c(d\vec{a}) = (cd)\vec{a}$

and satisfies the two distributive laws

$$c(\vec{\alpha}+\vec{\beta}) = c\vec{\alpha} + c\vec{\beta} \quad \text{and} \quad (c+d)\vec{\alpha} = c\vec{\alpha} + d\vec{\alpha}.$$

The zero Vector

The vector with all components 0 is called the zero vector and is denoted by $\vec{0}$.

It has the property that $\vec{\alpha}+\vec{0} = \vec{\alpha}$ for every vector $\vec{\alpha}$.

Note: Subtraction follows from the definition of vector addition and scalar multiplication.

For example, if $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$, then

$$\vec{v}-\vec{w} = \vec{v} + (-1)\vec{w} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} -w_1 \\ -w_2 \end{pmatrix} = \begin{pmatrix} v_1-w_1 \\ v_2-w_2 \end{pmatrix}.$$

Linear Combinations

Consider two vectors $\vec{v}$ and $\vec{w}$ and scalars $c, d \in \mathbb{R}$. (i.e., c and d are real numbers)

Then $c\vec{v}+d\vec{w}$ is a linear combination of $\vec{v}$ and $\vec{w}$.

For example, $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$$\underbrace{2\vec{v}-\vec{w}}_{\text{linear combination of } \vec{v} \text{ and } \vec{w}} = 2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+3 \\ 4-1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$$

Some important linear combinations

Sum of vectors: $\vec{v} + \vec{w}$

difference of vectors: $\vec{v} - \vec{w}$

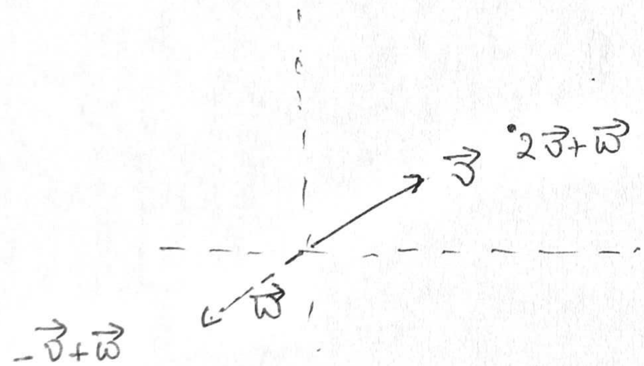
zero vector: $0\vec{v} + 0\vec{w}$

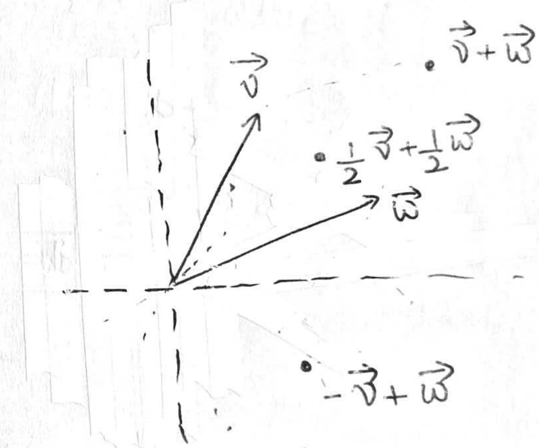
vector in the direction of $\vec{w}$: $0\vec{v} + d\vec{w}$

Linear combinations are at the heart of linear algebra!

Consider the set of all possible linear combinations of $\vec{v}$ and $\vec{w}$.

ie, c and d can be any real number. What do we get?





all linear combinations lie in a line

* see note at end of notes

all possible linear combinations make up the two-dimensional plane.

* If $\vec{v} = \vec{w} = (0,0)$ then every linear combination is the zero vector!

Now consider vectors in 3 dimensional space

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$

for typical non-zero (components chosen at random)

then,
- all combinations $c\vec{u}$ can fill a line
- all combinations $c\vec{u} + d\vec{v}$ can fill a plane
- — " — $c\vec{u} + d\vec{v} + e\vec{w}$ can fill three-dimensional space