Elimination, Inverse Matrices, LU factorization

① Compute A^{-1} if $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. Use the inverse to find the solution to $A\vec{x} = \vec{b}$ where $\vec{b} = (36, -36, 24)$.

② Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

(i) For what values of a and b will the system have infinitely many solutions?

(ii) — — have no solution?

③ Write down the LU factorization of $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$

Vector Spaces / Subspaces

① Determine if the following are subspaces:

(i) $\{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 0 \}$ (of $\mathbb{R}^2$)

(ii) $\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = x_1 + x_2 \}$ (of $\mathbb{R}^3$)

① Compute A^{-1} if $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. Use the inverse to find the solution to $A\vec{x} = \vec{b}$ where $\vec{b} = (36, -36, 24)$.

Performing elimination on $[A \,\vdots\, I]$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right]$$

(upper triangular form; continue with Gauss-Jordan)

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right]$$

$$\longrightarrow \Bigg[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \Bigg|\, \underbrace{\begin{array}{ccc} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}}_{A^{-1}} \Bigg]$$

$$\boxed{A^{-1} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}}$$

$A\vec{x} = \vec{b}$ solution

$$\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 36 \\ -36 \\ 24 \end{bmatrix}$$

$$\boxed{\vec{x} = \begin{bmatrix} 12 \\ 12 \\ -8 \end{bmatrix}}$$

② Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right]$$

(i) For what values of a and b will the system have infinitely many solutions?

(ii) — " — — " — have no solution?

Performing elimination on the augmented matrix

$$\left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-4 \end{array}\right]$$

(i) When $\boxed{a=5,\ b=4}$ the augmented matrix becomes $\left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

there are $\boxed{\text{infinitely many solutions}}$

(ii) When $\boxed{a=5,\ b\neq 4}$ the augmented matrix is $\left[\begin{array}{ccc:c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & c \end{array}\right]$ where $c \neq 0$

this system has $\boxed{\text{no solutions}}$

③ Write down the LU factorization of $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$

We perform elimination with elementary / elimination matrices

With $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, we have $E_1 A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 4 & -1 & 9 \end{bmatrix}$

With $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, we have $E_2(E_1 A) = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix}$

With $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}$, we have $E_3(E_2 E_1 A) = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = U$ (upper triangular)

$(E_3 E_2 E_1) A = U \implies A = (E_3 E_2 E_1)^{-1} U = (E_1^{-1} E_2^{-1} E_3^{-1}) U = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$ (lower triangular)

Hence $\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}}$

$\boxed{\text{Vector Spaces / Subspaces}}$

① Determine if the following are subspaces: (i) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$ (of $\mathbb{R}^2$)

(ii) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = x_1 + x_2\}$ (of $\mathbb{R}^3$)

(i) $\boxed{\text{NOT}}$ a subspace of $\mathbb{R}^2$

Counter-example: Let $A = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 x_2 = 0\}$

$(1,0), (0,1) \in A$. However $(1,0) + (0,1) = (1,1) \notin A$

closure of vector addition is not satisfied.

(ii) $\boxed{\text{YES,}}$ a subspace of $\mathbb{R}^3$ Let $B = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 = x_1 + x_2\}$

We show closure of vector addition, scalar multiplication

Vector Addition: Let $(x_1, x_2, x_3), (y_1, y_2, y_3) \in B$. Then $x_3 = x_1 + x_2$ —— ⓐ
(arbitrary)

$y_3 = y_1 + y_2$ —— ⓑ

$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

from ⓐ, ⓑ $(x_3 + y_3) = (x_1 + y_1) + (x_2 + y_2)$

$\Rightarrow (x_1, x_2, x_3) + (y_1, y_2, y_3) \in B$. Therefore vector addition is closed.

Scalar Multiplication Let $(x_1, x_2, x_3) \in B$ and $\lambda \in \mathbb{R}$

Then $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3) \in B$

since $\lambda x_3 = \lambda(x_1 + x_2)$ (from ⓐ)

$= (\lambda x_1) + (\lambda x_2)$

Hence we have closure under scalar multiplication.

Fundamental Subspaces, Bases, Dimension

① Determine the nullspace of $A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$

② Let $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$

(i) Are $\vec{x}_1, \vec{x}_2$ linearly independent?

(ii) Are $\vec{x}_1, \vec{x}_2, \vec{x}_3$ linearly independent?

③ Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$. Find bases for $N(A)$, $C(A)$, $C(A^T)$ and $N(A^T)$. What are the corresponding dimensions?

④ Find all possible solutions / complete solution to $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$

Determinants look at questions on quiz, midterm 2

① Determine the nullspace of $A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -4 & 6 & 3 \end{bmatrix}$

We find the reduced row echelon form

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ -2 & -4 & 6 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R$$

reduced row echelon form

pivot col pivot col

$$N(A) = \{\vec{x} \in \mathbb{R}^4 \mid A\vec{x} = \vec{0}\} = \{\vec{x} \in \mathbb{R}^4 \mid R\vec{x} = \vec{0}\}$$

$$R\vec{x} = \vec{0} \Rightarrow \left.\begin{aligned} x_1 + 2x_2 - 3x_3 &= 0 \\ x_4 &= 0 \end{aligned}\right\}$$ x_2, x_3 are free variables

Special solutions

$\vec{s}_1$: choose $x_2 = 1$, $x_3 = 0$, we get $x_4 = 0$, $x_1 = -2$ -or- $\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\vec{s}_2$: choose $x_2 = 0$, $x_3 = 1$, we get $x_4 = 0$, $x_1 = 3$ -or- $\vec{s}_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

$$N(A) = \text{span}\{\vec{s}_1, \vec{s}_2\} = \boxed{\text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}}$$

② Let $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$

(i) Are $\vec{x}_1, \vec{x}_2$ linearly independent?

(ii) Are $\vec{x}_1, \vec{x}_2, \vec{x}_3$ linearly independent?

(i) Let $A_1 = [\vec{x}_1 \ \vec{x}_2] = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 3 & 4 \end{bmatrix}$

perform elimination to find rank(A)

$$\begin{bmatrix} 2 & 3 \\ 1 & -1 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 \\ 0 & -5/2 \\ 0 & -1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 \\ 0 & -5/2 \\ 0 & 0 \end{bmatrix}$$

\# pivots = 2

$\Rightarrow$ rank (A) = 2

(full column rank)

Hence $\vec{x}_1, \vec{x}_2$ are $\boxed{\text{linearly independent}}$

(ii) Let $A_2 = [\vec{x}_1 \ \vec{x}_2 \ \vec{x}_3] = \begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{bmatrix}$

perform elimination to find rank(A)

$$\begin{bmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -5/2 & 5 \\ 0 & -1/2 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 3 & 2 \\ 0 & -5/2 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

\# pivots = 2

$\Rightarrow$ rank (A) = 2

rank (A) < # columns

Hence $\vec{x}_1, \vec{x}_2, \vec{x}_3$ are $\boxed{\text{NOT linearly independent}}$

(Alternatively $-4\vec{x}_1 + 2\vec{x}_2 + \vec{x}_3 = \vec{0}$

Hence $\vec{x}_1, \vec{x}_2, \vec{x}_3$ not linearly independent)

(3) Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 4 \end{bmatrix}$. Find bases for $N(A)$, $C(A)$, $C(A^T)$ and $N(A^T)$. What are the corresponding dimensions?

Perform elimination on augmented matrix $[A \,\vdots\, \vec{b}\,]$ where $\vec{b} = (b_1, b_2, b_3)$

$$\left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 1 & 3 & 4 & b_3 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 2 & 2 & b_3 - b_1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & -b_1 - 2b_2 + b_3 \end{array}\right]$$ (upper triangular)

$$\longrightarrow \left[\begin{array}{ccc:c} 1 & 0 & 1 & b_1 - b_2 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & -b_1 - 2b_2 + b_3 \end{array}\right]$$ (reduced row echelon form, R)

pivot cols (columns 1 and 2) free col (column 3)

$\boxed{C(A)}$

cols (1) and (2) are pivot cols

$C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}$ (from orig. matrix) Basis for $C(A)$ $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \right\}$

$\dim C(A) = 2$

$\boxed{C(A^T)}$

rows (1) and (2) are pivot rows

$C(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$ (from rref(A)) Basis for $C(A^T)$ $= \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\dim C(A^T) = 2$

$\boxed{N(A^T)}$

cfs of $\vec{b}$ on last row of $[R \,\vdots\, \vec{b}']$

$N(A^T) = \text{span}\left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$ Basis for $N(A^T)$ $= \left\{ \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \right\}$ $\dim N(A^T) = 1$

$\boxed{N(A)}$

$N(A) = \{ x \in \mathbb{R}^3 \mid A\vec{x} = \vec{0} \} = \{ \vec{x} \in \mathbb{R}^3 \mid R\vec{x} = \vec{0} \}$

1 free col $\Rightarrow$ 1 free var / 1 special sol^n.

(Set $\vec{x}_3 = 1$)

$\vec{s}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ $N(A) = \text{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Basis for $N(A)$ $= \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$ $\dim N(A) = 1$

alternative to obtain $\vec{s}_1$

$R\vec{x} = \vec{0} \Rightarrow \left.\begin{aligned} x_1 \quad + x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}\right\}$ $\xrightarrow[x_3 = 1 \text{ (free var)}]{\text{choose}}$ $x_1 = -1$, $x_2 = -1$, $\vec{s}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

Note: (alternate way to find $N(A^T)$)

$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ perform elimination to find rref $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 2 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

rref

free col.

Special sol^n $\vec{s}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

(4) Find all possible solutions / complete solution to $\begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 8 & 12 \\ 3 & 6 & 7 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 6 \\ -6 \end{bmatrix}$

let's perform elimination on $[A \vdots \vec{b}]$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 2 & 4 & 8 & 12 & 6 \\ 3 & 6 & 7 & 13 & -6 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & -2 & -2 & -6 \end{array}\right] \longrightarrow \underbrace{\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{(upper triangular)}}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 2 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 5 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 2 & -9 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

pivot cols

(reduced row echolon form R) $[R \vdots \vec{d}]$

Special sol^n's x_2, x_4 are free vars. ; solve $R\vec{x} = \vec{0}$

to find $\vec{s}_1$, choose $x_2 = 1, x_4 = 0$

$$\vec{s}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

to find $\vec{s}_2$, Choose $x_2 = 0, x_4 = 1$

$$\vec{s}_2 = \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Nullspace sol^n

$$\vec{x}_n = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

where $x_2, x_4 \in \mathbb{R}$

Particular sol^n

obtained by setting $x_2 = x_4 = 0$

solve $R\vec{x} = \vec{d}$

$$\vec{x}_p = \begin{pmatrix} -9 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Complete sol^n $\vec{x} = \vec{x}_p + \vec{x}_n$

$$\boxed{\vec{x} = \begin{pmatrix} -9 \\ 0 \\ 3 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ -1 \\ 1 \end{pmatrix}}$$