Eigenvalues and Eigenvectors

① Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Find the eigenvalues of A. Calculate the eigenvectors of A for the two largest eigenvalues

② Factor the matrix A into a product XDX^{-1} where D is diagonal $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

③ Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$ (i) How many negative eigenvalues does A have?

(ii) Calculate the product of the eigenvalues of A

(iii) Calculate the sum of the eigenvalues of A

(iv) Suppose you are given 3 eigenvectors of A. Explain how you find the fourth eigenvector without knowing any of A's eigenvalues.

④ Compute $u_{20} = A^{20} u_0$ starting with $A = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$ and $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

⑤ If you transpose $S^{-1} A S = \Lambda$, we get the eigenvalues of $A^T =$ __________

the eigenvectors of $A^T =$ __________

① Let $A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}$. Find the eigenvalues of A. Calculate the eigenvectors of A for the two largest eigenvalues

Eigenvalues $\quad |A - \lambda I| = 0 \implies \begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0$

$$\Rightarrow (1-\lambda)(-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \underline{\lambda = 0 \text{ or } \lambda = 1 \text{ or } \lambda = 2}$$

Eigenvectors $\boxed{\lambda = 1}$ Solve $(A - \lambda I)\vec{x} = \vec{0}$

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} -x_1 - x_2 + x_3 = 0 \\ x_1 + x_3 = 0 \end{array}\right\}$$

choosing $x_3 = 1$, (free var) we get $\boxed{\vec{x} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$

$\lambda = 2$ Solve $(A - 2I)\vec{y} = \vec{0}$

$$\begin{pmatrix} -1 & 0 & 0 \\ -1 & -2 & 1 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \left.\begin{array}{l} -y_1 = 0 \\ -y_1 - 2y_2 + y_3 = 0 \\ y_1 = 0 \end{array}\right\}$$

$\Rightarrow y_1 = 0$; choose $y_3 = 1$, (free var) we get $\boxed{\vec{y} = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}}$

(2) Factor the matrix A into a product XDX^{-1} where D is diagonal $A = \begin{bmatrix} 2 & -3 \\ 2 & -5 \end{bmatrix}$

Eigenvalues $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(-5-\lambda) + 6 = 0$

$\Rightarrow \lambda^2 + 3\lambda - 4 = 0$

$\Rightarrow (\lambda + 4)(\lambda - 1) = 0$

$\Rightarrow \boxed{\lambda = -4 \text{ or } \lambda = 1}$

Eigenvectors $\boxed{\lambda = -4}$ $(A + 4I)\vec{x} = \vec{0}$ $\begin{pmatrix} 6 & -3 \\ 2 & -1 \end{pmatrix} \vec{x} = \vec{0} \Rightarrow \boxed{\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ is an eigenvector

$\lambda = 1$ $(A - \lambda I)\vec{y} = \vec{0}$ $\begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix} \vec{y} = \vec{0} \Rightarrow \boxed{\vec{y} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$ is an eigenvector

Let $X = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ then $X^{-1} = \frac{1}{-5} \begin{pmatrix} 1 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{pmatrix}$

$D = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix}$. Then $A = \begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/5 & 3/5 \\ 2/5 & -1/5 \end{pmatrix}$

We can verify that $A = XDX^{-1}$ and $D = X^{-1}AX$.

(3) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix}$

(i) How many negative eigenvalues does A have?

(ii) Calculate the product of the eigenvalues of A

(iii) Calculate the sum of the eigenvalues of A

(iv) Suppose you are given 3 eigenvectors of A. Explain how you find the fourth eigenvector without knowing any of A's eigenvalues.

Note that A is symmetric. We use elimination to reduce to upper triangular form. Then

\# negative eigenvalues = \# negative pivots

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 2 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & \frac{1}{2} & 2 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

(i) 1 negative pivot $\Rightarrow$ 1 negative eigenvalue.

(ii) Product of eigenvalues $= \prod_{i=1}^{4} \lambda_i = |A| =$ product of pivots in triangular form $= -3$

(iii) Sum of eigenvalues $= \sum_{i=1}^{4} \lambda_i =$ trace $(A) = 10$

(iv) Since A is symmetric, its eigenvectors are orthogonal. The fourth e-vector is the vector perpendicular to the given three e-vectors.

④ Compute $u_{20} = A^{20} u_0$ starting with $A = \begin{bmatrix} 0.7 & 0.5 \\ 0.3 & 0.5 \end{bmatrix}$ and $u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Use same procedure as ③ to find e-values and e-vectors (Note: A is a Markov matrix $\Rightarrow \lambda_1 = 1$)

We get e-values $\boxed{\lambda_1 = 1 \quad \lambda_2 = 0.2}$

e-vectors $\boxed{\vec{x}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$

Distinct e-values $\Longrightarrow$ (diagonalization)

$$A = \underbrace{\begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1/8 & 1/8 \\ -3/8 & 5/8 \end{pmatrix}}_{X^{-1}}$$

$$\Rightarrow u_{20} = A^{20} u_0 = \frac{1}{8} \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1^{20} & 0 \\ 0 & 0.2^{20} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0.2^{20} \end{pmatrix} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \underbrace{-3(0.2^{20})}_{\approx 0} \end{pmatrix}$$

$$\boxed{u_{20} = \frac{1}{8} \begin{pmatrix} 5 + 3(0.2^{20}) \\ 3 - 3(0.2^{20}) \end{pmatrix}}$$

note: $u_{20} \approx \begin{pmatrix} 5/8 \\ 3/8 \end{pmatrix}$

⑤ If you transpose $S^{-1} A S = \Lambda$, we get the eigenvalues of A^T = same as A

the eigenvectors of A^T = are columns of $(S^T)^{-1}$

Note: $(S^{-1} A S)^T = \Lambda^T = \Lambda$ (diagonal) $\Rightarrow S^T A^T (S^{-1})^T = \Lambda$

$\Rightarrow A^T = (S^T)^{-1} \Lambda S^T$

QR, Least-Squares, Projections

① Given $A = \begin{pmatrix} 2 & -3 & -5 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \\ 2 & 1 & 4 \end{pmatrix}$,

(i) apply Gram-Schmidt to determine an orthonormal basis for $C(A)$ and a QR factorization of A

(ii) Use the QR factorization to determine the least-squares solution of $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

② Find the curve $y = a + 2^x b$ which gives the best least-squares fit to the points $(x, y) = (0,6), (1,4), (2,0)$

③ Let $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

(i) Find the projection $\vec{p}$ of $\vec{x}$ onto $\vec{y}$.

(ii) Verify that $\vec{x} - \vec{p}$ is orthogonal to $\vec{y}$.

④ Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}$. Find the projection of $\vec{b}$ onto $C(A)$

① Given $A = \begin{array}{c} \begin{pmatrix} 2 & -3 & -5 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \\ 2 & 1 & 4 \end{pmatrix} \\ \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array} \end{array}$, (i) apply Gram-Schmidt to determine an orthonormal basis for $C(A)$ and a QR factorization of A

(ii) Use the QR factorization to determine the least-squares solution of $A\vec{x} = \vec{b}$ where $\vec{b} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 2 \end{pmatrix}$

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Step 1

$r_{11} = \|\vec{a}_1\| = \sqrt{2^2+2^2+2^2+2^2} = 4$

$\vec{q}_1 = \dfrac{\vec{a}_1}{\|\vec{a}_1\|} = \boxed{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right)}$

Step 2

$r_{12} = (\vec{a}_2 \cdot \vec{q}_1) = (-3, 1, -3, 1) \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = -2$

Let $\vec{v}_2 = \vec{a}_2 - (\vec{a}_2 \cdot \vec{q}_1)\, \vec{q}_1$

$= \begin{pmatrix} -3 \\ 1 \\ -3 \\ 1 \end{pmatrix} - (-2) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \boxed{\begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix}}$

$r_{22} = \|\vec{v}_2\| = 4$

$\vec{q}_2 = \dfrac{\vec{v}_2}{\|\vec{v}_2\|} = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix}$

Step 3

$r_{13} = (\vec{a}_3 \cdot \vec{q}_1) = (-5, -2, 1, 4) \cdot \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) = -1$

$r_{23} = (\vec{a}_3 \cdot \vec{q}_2) = (-5, -2, 1, 4) \cdot \left(-\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}\right) = 3$

Let $\vec{v}_3 = \vec{a}_3 - (\vec{a}_3 \cdot \vec{q}_1)\, \vec{q}_1 - (\vec{a}_3 \cdot \vec{q}_2)\, \vec{q}_2$

$= \begin{pmatrix} -5 \\ -2 \\ 1 \\ 4 \end{pmatrix} - (-1) \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} - 3 \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \\ 3 \end{pmatrix}$

$r_{33} = \|\vec{v}_3\| = \sqrt{(-3)^2 + (-3)^2 + 3^2 + 3^2} = 6$

$\vec{q}_3 = \dfrac{\vec{v}_3}{\|\vec{v}_3\|} = \boxed{\begin{pmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$

QR Decomposition

$Q = \frac{1}{2} \begin{array}{c} \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ \begin{array}{ccc} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \end{array} \end{array}$ $\quad R = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$

Least-Squares Solution

$$A^T A\vec{x} = A^T\vec{b}$$

using $A = QR$, we get

$$(QR)^T(QR)\vec{x} = (QR)^T\vec{b}$$

$$\Rightarrow \quad R^T\underbrace{(Q^TQ)}_{=I}R\vec{x} = R^TQ^T\vec{b}$$

$$\Rightarrow \quad R\vec{x} = Q^T\vec{b}$$

QR Decomposition

$$A = \begin{pmatrix} 2 & -3 & -5 \\ 2 & 1 & -2 \\ 2 & -3 & 1 \\ 2 & 1 & 4 \end{pmatrix} \qquad \begin{array}{ccc} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{array}$$

$$Q = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array}$$

$$R = \begin{bmatrix} 4 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

$$\vec{b} = (-3, 1, 0, 2)$$

$$Q^T\vec{b} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \end{bmatrix}\begin{pmatrix} -3 \\ 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$R\vec{x} = Q^T\vec{b} \iff \begin{bmatrix} 4 & -2 & -1 \\ 0 & 4 & 3 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

(Solving by back substitution)

$$x_3 = 1/3$$

$$x_2 = \frac{3 - 3(1/3)}{4} = 1/2$$

$$x_1 = \frac{1/3 + 2(1/2)}{4} = 1/3$$

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$$\boxed{\vec{x} = \begin{pmatrix} 1/3 \\ 1/2 \\ 1/3 \end{pmatrix}}$$

② Find the curve $y = a + 2^x b$ which gives the best least-squares fit to the points $(x, y) = (0,6), (1,4), (2,0)$

If the curve went through all 3 points

$$\begin{array}{ll} (x=0) & a + b = 6 \\ (x=1) & a + 2b = 4 \\ (x=2) & a + 4b = 0 \end{array} \Bigg\} \quad \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}}_{\vec{y}}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

Least-squares solution

$$A^T A \vec{x} = A^T \vec{b}$$

$$\Updownarrow$$

$$\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$$

Least-squares parameter fit: $a = 8,\ b = -2$

③ Let $\vec{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

(i) Find the projection $\vec{p}$ of $\vec{x}$ onto $\vec{y}$.

(ii) Verify that $\vec{x} - \vec{p}$ is orthogonal to $\vec{y}$.

(i) $\vec{p} = \dfrac{(\vec{x} \cdot \vec{y})}{(\vec{y} \cdot \vec{y})} \vec{y} = \dfrac{3}{9} \vec{y} = \dfrac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

(ii) $\vec{x} - \vec{p} = \dfrac{1}{3} \begin{pmatrix} 5 \\ 2 \\ 4 \\ 6 \end{pmatrix}$ $\quad (\vec{x} - \vec{p}) \cdot \vec{y} = -\dfrac{10}{3} + \dfrac{2}{3} + \dfrac{8}{3} + 0 = 0$

#4

$$\boxed{A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 12 \\ 6 \\ 18 \end{pmatrix}}$$

Projection of $\vec{b}$ onto $C(A)$ $\qquad \vec{p} = A(A^TA)^{-1}A^T\vec{b}$

$$A^TA = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix} \qquad (A^TA)^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -5 \\ -5 & 9 \end{pmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix} = \begin{bmatrix} 66 \\ 36 \end{bmatrix} = 6\begin{bmatrix} 11 \\ 6 \end{bmatrix}$$

$$\vec{p} = A(A^TA)^{-1}A^T\vec{b} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 3 & -5 \\ -5 & 9 \end{bmatrix}\right)\left(6\begin{bmatrix} 11 \\ 6 \end{bmatrix}\right)$$

not asked in problem; provided here for illustration

$$= 3\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$= 3\begin{bmatrix} 5 \\ 2 \\ 5 \end{bmatrix} = \begin{pmatrix} 15 \\ 6 \\ 15 \end{pmatrix}$$