Week 3 – Worksheet – MTH 305 (Spring 2017)

(1) Zeno's racecourse paradox: Greek philosopher Zeno precipitated a mathematical crisis by setting forth paradoxes such as the following:

A runner can never reach the end of a racecourse because he must cover half of any distance before he covers the whole. That is to say, having covered the first half he still has the second half before him. When half of this is covered, one-fourth yet remains. When half of this one-fourth is covered, there remains one-eighth, and so on, *ad infinitum*

(a) Discuss and deliberate the merits of the above argument. Can you justify whether the above statement is true or false?

(b) You are now going to formulate the problem mathematically. Suppose the runner travels at a *constant speed* and suppose it takes him/her T minutes to cover the first half of the course. How long will it take him/her to cover the next quarter of the course? How about the next eighth? And, in general, how about the portion of the racecourse from $\dfrac{1}{2^n}$ to $\dfrac{1}{2^{n+1}}$?

(c) Can you now write down an expression for the total time taken for the runner to cover the racecourse? Can you relate this expression to the topics on sequences/series you have discussed in class? What type of a sequence or series is this?

(2) You will now consider a variant of the above problem. Instead of assuming that the speed of the runner is constant, suppose that his/her speed gradually decreases in such a way that he/she requires T minutes to cover the first half of the course, $\dfrac{T}{2}$ minutes to cover the next quarter of the course, $\dfrac{T}{3}$ minutes to cover the next eighth of the course, and in general $\dfrac{T}{n}$ minutes to cover the successive $\dfrac{1}{2^{n-1}} \to \dfrac{1}{2^n}$ portion of the course.

(a) Write down an expression for the total time taken to cover the course. (this should be an infinite series)

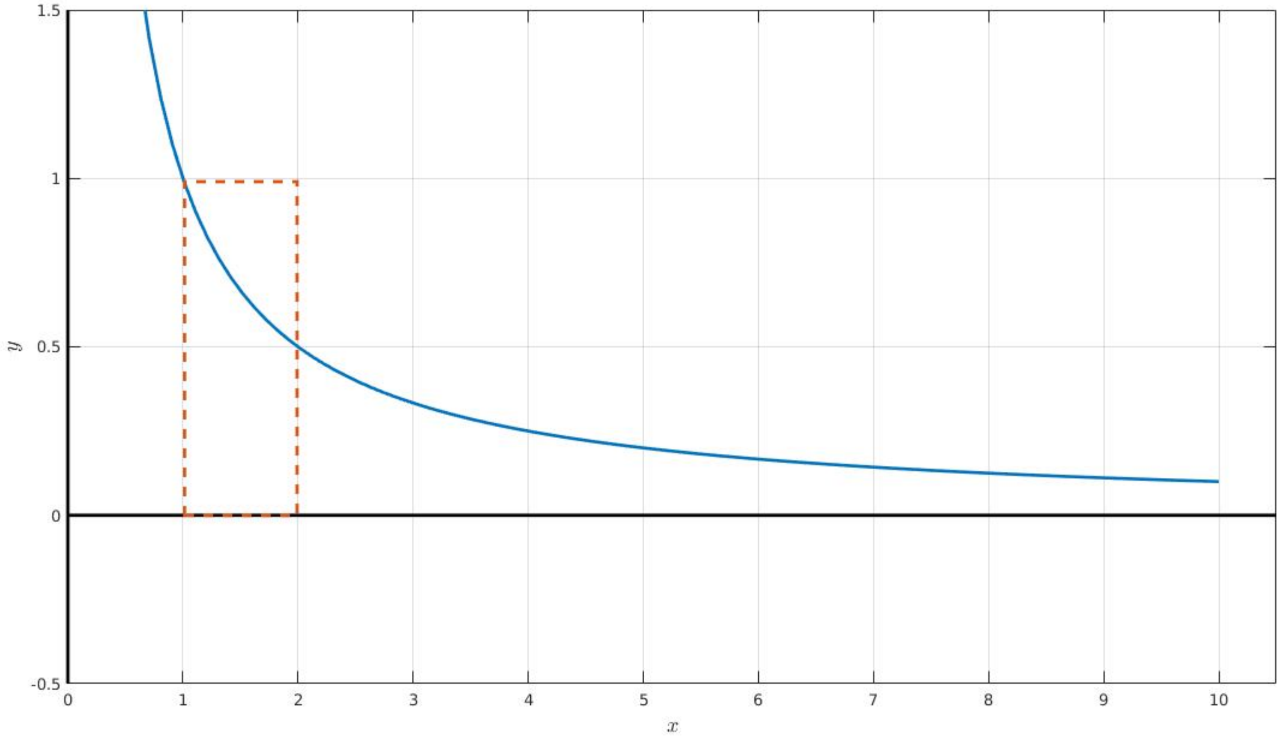
(b) Can you decide if this series is convergent? (i.e., does the runner cover the racecourse in a finite amount of time?)

(c) To assist in your reasoning, consider the plot in the following page. This is a graph of the function $y = \dfrac{1}{x}$. Draw rectangles of width 1 unit and heights $1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}$ respectively; the first rectangle (of height 1 unit) has been drawn for illustration. The second rectangle of height $\dfrac{1}{2}$ unit should lie immediately to the right of this rectangle.

(i) What is the sum of the areas of the rectangles?

(ii) Consider the area under the graph and the x-axis between $x = 1$ and $x = 10$. Shade this area. Is this area less than or greater than the sum of the areas of the rectangles? Can you provide an exact value for this area? Does this help answer the question about series convergence?

(iii) Imagine extending the plot to $x = 20, 100, \ldots$. The width of the rectangles remains the same, but the number of rectangles, n, increases. Discuss what happens as $n \to \infty$.



(3) Consider the infinite series

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots,$$

where $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$.

(a) Evaluate the first 5 partial sums, s_n, of the series (use a calculator if necessary)

(b) Consider the sequence of partial sums. Is this sequence an increasing or decreasing sequence?

(c) Can you guess the value of $\lim_{n\to\infty} s_n$?

<u>Some review exercises</u>

(4) Decide if the series converges or diverges. If the series converges, compute its sum.

(a) $\frac{5}{6} + \frac{25}{36} + \frac{125}{216} + \dots$

(b) $\frac{1}{3} - \frac{1}{3} + \frac{1}{3} - \frac{1}{3} + \dots$

(c) $2 + \frac{5}{2} + 3 + \frac{7}{2} + 4 + \frac{9}{2} + \dots$

(5) Compute the sum without using a calculator.

(a) $S = 2 + 8 + 14 + \dots + 638$

(b) $S = 10\sqrt{5} + 50 + 50\sqrt{5} + \dots + 3,906,250$

(c) $S = 1 + 3.5 + 6 + 8.5 + \dots + 101$

(6) If $\{x_n\}_n$ is an arithmetic sequence, compute the sum

$$\frac{1}{\sqrt{x_1}+\sqrt{x_2}} + \frac{1}{\sqrt{x_2}+\sqrt{x_3}} + \cdots \frac{1}{\sqrt{x_{n-1}}+\sqrt{x_n}}$$

in terms of x_1, the common difference d and n. All the terms in the sum are well-defined.

(7) If $\{y_n\}_n$ is a geometric sequence with common ratio r, compute the sum

$$S = y_1y_2y_3 + y_2y_3y_4 + \cdots + y_ny_{n+1}y_{n+2}$$

in terms of y_1 and r.

(8) If a is a non-zero real number, and $n \geq 2$, compute the sum

$$\left(a+\frac{1}{a}\right)^2 + \left(a^2+\frac{1}{a^2}\right)^2 + \cdots + \left(a^n+\frac{1}{a^n}\right)^2.$$