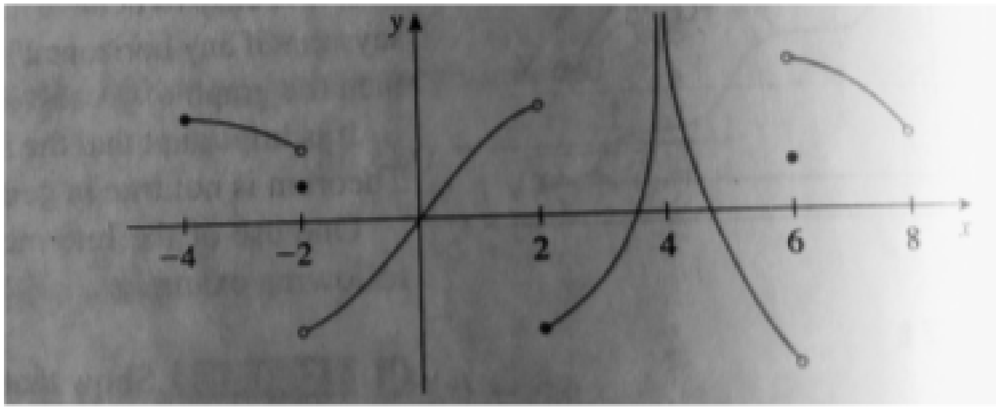
Selected Solutions - Week 8

(1) From the graph of g, state the intervals on which g is continuous.

#1



g is continuous on the following intervals

* $(-4,-2)$
* $(-2,2)$
* $(2,4)$
* $(4,6)$
* $(6,8)$

Note:

* g is continuous from the right at $x=-4$
* g is not continuous at $x=-2$ since $\lim_{x\to -2^-} g(x) \neq \lim_{x\to -2^+} g(x) \neq g(-2)$
* g is not continuous at $x=2$ since $\lim_{x\to 2^-} g(x) \neq \lim_{x\to 2^+} g(x)$
* g is not continuous at $x=4$ since $\lim_{x\to 4} g(x)$ does not exist
* g is not continuous at $x=6$ since $\lim_{x\to 6^-} g(x) \neq \lim_{x\to 6^+} g(x) \neq g(6)$

(2) Where are each of the following functions discontinuous?

#2

(a) $f(x) = \dfrac{x^2 - x - 12}{x - 4}$ at $x = 4$ since $f(4)$ not defined

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $x = 0$ since $\lim_{x \to 0} f(x)$ not defined

(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 12}{x - 4} & \text{if } x \neq 4 \\ 1 & \text{if } x = 4 \end{cases}$ at $x = 4$ since $\lim_{x \to 4} f(x) \neq f(4)$

(d) $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 2x^2 & \text{if } 0 \leq x \leq 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

At which of these numbers is f continuous from the right, from the left, or neither. Sketch the graph of f.

at $x = 0$ since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$ (i.e., $\lim_{x \to 0} f(x)$ does not exist)

and

at $x = 1$ since $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$ (i.e., $\lim_{x \to 1} f(x)$ does not exist)

Note: at $x = 0$, f is continuous from the right since $\lim_{x \to 0^+} f(x) = f(0) = 0$.

at $x = 1$, f is " " left since $\lim_{x \to 1^-} f(x) = f(1) = 2$.

#4 Consider the function

$$f(x) = \begin{cases} \dfrac{x^2+4x-5}{x-1} & \text{if } x \neq 1 \\ a & \text{if } x = 1 \end{cases}$$

Find the value of a so that f is continuous everywhere

SOLUTION

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2+4x-5}{x-1}$$

$$= \lim_{x\to 1} \frac{(x+5)\cancel{(x-1)}}{\cancel{x-1}}$$

$$= \lim_{x\to 1} x+5$$

$$= 6.$$

Hence, if $a=6$, then $\lim_{x\to 1} f(x) = f(1)$, i.e., f is continuous at $x=1$.

For all other values of x, we also see that f is continuous.

#5

$$f(x) = \begin{cases} Ax - B & \text{if } x \leq -1 \\ 2x^2 + 3Ax + B & \text{if } -1 < x \leq 1 \\ 4 & \text{if } x > 1 \end{cases}$$

Want: $f(x)$ continuous for all x.

hence, we want

(i) $\lim_{x \to -1} f(x) = f(-1) \Rightarrow \lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x)$

(ii) $\lim_{x \to 1} f(x) = f(1) \Rightarrow \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)$

We get

$A(-1) - B = 2(-1)^2 + 3A(-1) + B$ $\qquad$ $2(1)^2 + 3A(1) + B = 4$

or $\qquad$ or

$-A - B = 2 - 3A + B$ $\qquad$ $2 + 3A + B = 4$

or $\qquad$ or

$\boxed{2A - 2B = 2}$ $\qquad$ Solve these 2 eqns for A, B $\qquad$ $\boxed{3A + B = 2}$

should get $\boxed{A = \frac{3}{4}, \; B = -\frac{1}{4}}$

(#6) Use the intermediate value theorem to show that there is a root of the equation $x^4+x-3=0$ between 1 and 2.

Let $f(x) = x^4 + x - 3$.

We note that $f(1) = 1^2 + 1 - 3 = -1$ $\quad f(1) < 0$

$f(2) = 2^4 + 2 - 3 = 15$ $\quad f(2) > 0$

Now, from the intermediate value theorem,

there exists $c \in (1,2)$ such that $f(1) < f(c) < f(2)$

Letting $f(c) = 0$ (note that $-1 < 0 < 15$ holds) implies that c is a root of the equation $x^4 + x - 3$ and that this root lies between 1 and 2.

#7 Show that the function $f(x) = 1 - \sqrt{4-x^2}$ is continuous on the interval $[-2, 2]$.

If $-2 < a < 2$, we have (using limit properties)

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left[1 - \sqrt{4 - x^2} \right]$$

$$= 1 - \lim_{x \to a} \sqrt{4 - x^2}$$ (by property 9(b))

(see formula sheet)

$$= 1 - \sqrt{\lim_{x \to a} (4 - x^2)}$$ (by property 9(k))

$$= 1 - \sqrt{4 - a^2}$$ (by properties 9(b), 9(g), 9(h))

$$\lim_{x \to a} f(x) = f(a)$$

Thus, by definition, f is continuous at a if $-2 < a < 2$.

Similarly, we can show that $\lim_{x \to -2^+} f(x) = 1 = f(-2)$ and $\lim_{x \to 2^-} f(x) = 1 = f(2)$.

(f continuous from the right at $x = -2$) (f continuous from the left at $x = 2$)

Thus, by definition, f is continuous on $[-2, 2]$.

Note/ Recall

A function f is continuous on an interval if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.