#5 (a) Compute the sum without using a calculator.

$$S = 2 + 8 + 14 + \dots + 638$$

This is an arithmetic sequence.

First term, $a_1 = 2$ Common difference $d = 6$

(to find n) Since $a_n = a_1 + (n-1)d$, we have

$$638 = 2 + (n-1)6$$

$$\Rightarrow 636 = (n-1)6$$

$$\Rightarrow 106 = n-1$$

$$\Rightarrow \underline{n = 107}$$

Now, $S = \frac{n}{2}(a_1 + a_n)$

$$= \frac{107}{2}(2 + 638)$$

$$= \frac{107(640)}{2},$$

or, $\underline{S = 107(320)} = 34{,}240.$

Recall

n^{th} term

$$\boxed{a_n = a_1 + (n-1)d}$$

Sum to n terms

$$\boxed{S_n = \frac{n}{2}(a_1 + a_n)}$$

#5 (b) Compute the sum

$$S = 10\sqrt{5} + 50 + 50\sqrt{5} + \cdots + 3,906,250$$

this is a geometric series with common ratio $r = \sqrt{5}$.

Recall

n^{th} term

$$\boxed{a_n = a_1 r^{n-1}}$$

Sum to n terms

$$\boxed{S_n = \frac{a_1(1-r^n)}{1-r}}$$

to find n

$$a_n = a_1 r^{n-1}$$

$$3,906,250 = 10\sqrt{5}\,(\sqrt{5})^{n-1}$$

$$= 10\cancel{\sqrt{5}}\,\frac{(\sqrt{5})^n}{\cancel{\sqrt{5}}}$$

$$3,906,250 = 10(\sqrt{5})^n$$

(dividing by 10) $\quad 390,625 = (\sqrt{5})^n = 5^{n/2}$

Since $5^8 = 390,625$, we have $5^8 = 5^{n/2}$

$\Rightarrow 8 = \frac{n}{2}$, or, $\underline{n = 16}$.

Now, $S = \dfrac{a_1(1-r^n)}{1-r} = \dfrac{10\sqrt{5}\,(1-\sqrt{5}^{16})}{1-\sqrt{5}}$

#6 If $\{x_n\}_n$ is an arithmetic sequence, compute the sum

$$\frac{1}{\sqrt{x_1}+\sqrt{x_2}} + \frac{1}{\sqrt{x_2}+\sqrt{x_3}} + \cdots + \frac{1}{\sqrt{x_{n-1}}+\sqrt{x_n}}$$

in terms of x_1, the common difference d and n. All the terms in the sum are well-defined.

Let's rewrite the first term as follows

$$\frac{1}{\sqrt{x_1}+\sqrt{x_2}} = \frac{1}{\sqrt{x_1}+\sqrt{x_2}}\left(\frac{\sqrt{x_1}-\sqrt{x_2}}{\sqrt{x_1}-\sqrt{x_2}}\right)$$

$$= \frac{\sqrt{x_1}-\sqrt{x_2}}{(\sqrt{x_1})^2-(\sqrt{x_2})^2} = \frac{\sqrt{x_1}-\sqrt{x_2}}{x_1-x_2} = \frac{\sqrt{x_2}-\sqrt{x_1}}{x_2-x_1} \qquad (*)$$

(multiplying num! and den' by -1)

Now, since $\{x_n\}_n$ is an arithmetic sequence, we have $x_2 = x_1 + d$

$$\Rightarrow \underline{x_2 - x_1 = d}$$

Therefore, substituting in $(*)$, we obtain

$$\frac{1}{\sqrt{x_1}+\sqrt{x_2}} = \frac{\sqrt{x_2}-\sqrt{x_1}}{x_2-x_1} = \frac{\sqrt{x_2}-\sqrt{x_1}}{d} \qquad (a)$$

Following a similar procedure, we obtain

Let's rewrite the second term as follows

$$\frac{1}{\sqrt{x_2}+\sqrt{x_3}} = \frac{1}{\sqrt{x_2}+\sqrt{x_3}}\left(\frac{\sqrt{x_2}-\sqrt{x_3}}{\sqrt{x_2}-\sqrt{x_3}}\right)$$

$$= \frac{\sqrt{x_2}-\sqrt{x_3}}{(\sqrt{x_2})^2-(\sqrt{x_3})^2} = \frac{\sqrt{x_2}-\sqrt{x_3}}{x_2-x_3} = \frac{\sqrt{x_3}-\sqrt{x_2}}{x_3-x_2} \quad \text{——} \quad (**)$$

(multiplying num.r and den.r by −1)

Now, since $\{x_n\}_n$ is an arithmetic sequence, we have $x_3 = x_2 + d$

$$\Rightarrow \underline{x_3 - x_2 = d}$$

Therefore, substituting in (**), we obtain

$$\frac{1}{\sqrt{x_2}+\sqrt{x_3}} = \frac{\sqrt{x_3}-\sqrt{x_2}}{x_3-x_2} = \frac{\sqrt{x_3}-\sqrt{x_2}}{d} \quad \text{——} \quad (b)$$

Similarly, we obtain

$$\frac{1}{\sqrt{x_{n-1}}+\sqrt{x_n}} = \frac{\sqrt{x_n}-\sqrt{x_{n-1}}}{d} \quad \text{——} \quad (c)$$

Using ⓐ ⓑ and ⓒ, we have

$$\frac{1}{\sqrt{x_1}+\sqrt{x_2}} + \frac{1}{\sqrt{x_2}+\sqrt{x_3}} + \cdots + \frac{1}{\sqrt{x_{n-1}}+\sqrt{x_n}} = \frac{\sqrt{x_2}-\sqrt{x_1}}{d} + \frac{\sqrt{x_3}-\sqrt{x_2}}{d} + \cdots + \frac{\sqrt{x_n}-\sqrt{x_{n-1}}}{d}$$

$$= \frac{1}{d}\Big(\cancel{\sqrt{x_2}}-\sqrt{x_1} + \cancel{\sqrt{x_3}} - \cancel{\sqrt{x_2}} + \cancel{\sqrt{x_4}} - \cancel{\sqrt{x_3}} + \cdots$$

$$+ \cancel{\sqrt{x_{n-1}}} - \cancel{\sqrt{x_{n-2}}} + \sqrt{x_n} - \cancel{\sqrt{x_{n-1}}}\Big)$$

$$= \frac{-\sqrt{x_1}+\sqrt{x_n}}{d}$$

$$\boxed{\frac{1}{\sqrt{x_1}+\sqrt{x_2}} + \frac{1}{\sqrt{x_2}+\sqrt{x_3}} + \cdots + \frac{1}{\sqrt{x_{n-1}}+\sqrt{x_n}} = \frac{-\sqrt{x_1}+\sqrt{x_1+(n-1)d}}{d}}$$

#7 If $\{y_n\}_n$ is a geometric sequence with common ratio r, compute the sum

$$S = y_1 y_2 y_3 + y_2 y_3 y_4 + \dots + y_n y_{n+1} y_{n+2}$$

in terms of y_1 and r.

Given: $\{y_n\}_n$ is a geometric sequence with common ratio r.

Let's rewrite the first term of S as

$$y_1 y_2 y_3 = y_1 (y_1 r)(y_1 r^2) = y_1^3 r^3$$

(Since $\{y_n\}_n$ is a geometric sequence)

Similarly,

$$y_2 y_3 y_4 = (y_1 r)(y_1 r^2)(y_1 r^3) = y_1^3 r^6$$

$$\vdots$$

$$y_n y_{n+1} y_{n+2} = (y_1 r^{n-1})(y_1 r^n)(y_1 r^{n+1}) = y_1^3 r^{3n}$$

writing all terms of S using only y_1 and r

Now, observe that

$\boxed{\{y_n y_{n+1} y_{n+2}\}_n}$ is a geometric sequence with first term $\boxed{y_1^3 r^3}$ and common ratio $\boxed{r^3}$

Then, we have

$$S = S_n = \frac{a_1 (1-r^n)}{(1-r)} = \frac{(y_1^3 r^3)(1-r^{3n})}{(1-r^3)}.$$

#8 If a is a nonzero number and $n \geq 2$, compute the sum

$$S = \left(a+\frac{1}{a}\right)^2 + \left(a^2+\frac{1}{a^2}\right)^2 + \cdots\cdots + \left(a^n+\frac{1}{a^n}\right)^2$$

Expanding the squares,

$$S = \left(a^2 + 2\cdot\cancel{a}\cdot\frac{1}{\cancel{a}} + \frac{1}{a^2}\right) + \left(a^4 + 2\cdot\cancel{a^2}\cdot\frac{1}{\cancel{a^2}} + \frac{1}{a^4}\right) + \cdots\cdots + \left(a^{2n} + 2\cdot\cancel{a^n}\cdot\frac{1}{\cancel{a^n}} + \frac{1}{a^{2n}}\right)$$

$$= \left(a^2 + 2 + \frac{1}{a^2}\right) + \left(a^4 + 2 + \frac{1}{a^4}\right) + \cdots + \left(a^{2n} + 2 + \frac{1}{a^{2n}}\right)$$

(rearranging terms)

$$= \underbrace{\left(a^2 + a^4 + \cdots + a^{2n}\right)}_{\text{Geometric series with } r = a^2} + \underbrace{\left(2 + 2 + \cdots + 2\right)}_{n \text{ times} = 2n} + \underbrace{\left(\frac{1}{a^2} + \frac{1}{a^4} + \cdots + \frac{1}{a^{2n}}\right)}_{\text{Geometric series with common ratio } \frac{1}{a^2}}$$

(using $S_n = \dfrac{a_1(1-r^n)}{1-r}$)

$$= \frac{a^2(1-a^{2n})}{1-a^2} + 2n + \frac{\frac{1}{a^2}\left(1-\frac{1}{a^{2n}}\right)}{1-\frac{1}{a^2}}$$

$$\frac{\frac{1}{a^2}\left(1-\frac{1}{a^{2n}}\right)}{1-\frac{1}{a^2}} = \frac{\frac{1}{\cancel{a^2}}\left(\frac{a^{2n}-1}{a^{2n}}\right)}{\frac{a^2-1}{\cancel{a^2}}} = \frac{1}{a^{2n}}\left(\frac{a^{2n}-1}{a^2-1}\right) = \frac{1}{a^{2n}}\left(\frac{1-a^{2n}}{1-a^2}\right)$$

$$= \frac{a^2(1-a^{2n})}{1-a^2} + 2n + \frac{1}{a^{2n}}\frac{(1-a^{2n})}{1-a^2}$$

or,

$$\boxed{S = \left(\frac{1-a^{2n}}{1-a^2}\right)\left(a^2 + \frac{1}{a^{2n}}\right) + 2n \qquad \text{when } a \neq \pm 1.}$$

If $a = \pm 1$, then

$$\left(a+\frac{1}{a}\right)^2 = \left(1+\frac{1}{1}\right)^2 = 2^2 = 4$$

$$\left(a^2+\frac{1}{a^2}\right)^2 = \left(1+\frac{1}{1}\right)^2 = 2^2 = 4$$

$$\vdots$$

$$\left(a^n+\frac{1}{a^n}\right)^2 = \left(1+\frac{1}{1}\right)^2 = 2^2 = 4$$

$$\boxed{S = \underbrace{4 + 4 + \cdots + 4}_{n \text{ times}} = 4n \qquad \text{when } a = \pm 1}$$