#3 Compute the derivative of

(e) $h(t) = 3\sqrt{3t^2+2t+1}$

Using derivative rule, we have

$$\frac{dh}{dt} = 3\,\frac{d}{dt}\left(\sqrt{3t^2+2t+1}\right)$$ (constant multiple rule)

$$= 3\,\frac{\frac{d}{dt}\left(3t^2+2t+1\right)}{2\sqrt{3t^2+2t+1}}$$ (square root rule)

$$= \frac{3\left(3\frac{d}{dt}(t^2) + 2\frac{d}{dt}(t) + \frac{d}{dt}(1)\right)}{2\sqrt{3t^2+2t+1}}$$ (sum, constant mult. rules)

$$= \frac{3\left(6t + 2 + 0\right)}{2\sqrt{3t^2+2t+1}}$$ (power rule)

$$= \frac{18t+6}{2\sqrt{3t^2+2t+1}},$$ or $$\boxed{h'(t) = \frac{9t+3}{\sqrt{3t^2+2t+1}}}$$

#4 Find equations of the tangent line and normal line at the given point

(b) $y = x + 2\sqrt{x}$ at the point corresponding to $x=1$.

First, note that when $x=1$, $y = 1 + 2\sqrt{1} = 3$.

Therefore, we want to find tangent/normal lines passing through the point $\underline{(x_0, y_0) = (1, 3)}$

We also know that the slope of the tangent line to the graph of $f(x) = x + 2\sqrt{x}$ at $x=1$ is $m = f'(x)\Big|_{x=1} = 1 + 2 \cdot \frac{1}{2\sqrt{x}}\Big|_{x=1} = 1 + 1$ or $\underline{m = 2}$.

Using the point-slope form, eqn. of $\boxed{\text{tangent line}}$ is $y - 3 = 2(x-1)$ or $\boxed{2x - y + 1 = 0}$

Since the normal line is perpendicular to the tangent line,

$$m_{\text{tangent}} \cdot m_{\text{normal}} = -1, \quad \text{or} \quad m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}} = -\frac{1}{2}.$$

Eqn. of $\boxed{\text{normal line}}$ then is $y - 3 = -\frac{1}{2}(x-1)$ or $\boxed{x + 2y - 7 = 0}$

#5 Determine the intervals where the function is decreasing / increasing

(b) $g(x) = 3x^4 - 16x^3 + 18x^2$ over the interval $[-1, 4]$

We start by finding the derivative $g'(x)$

We have $g'(x) = 12x^3 - 48x^2 + 36x$

$= x(12x^2 - 48x + 36)$

$= x(12x - 36)(x-1) = 12x(x-1)(x-3)$ ← factorized form

Now, let us determine the sign of each of these terms (and hence, $g'(x)$) in the intervals $(-1,0), (0,1), (1,3), (3,4)$ ← these intervals are chosen based on the factorization for g': $(x-1)$ changes sign depending on if $x<1$ or $x>1$.

(Sign of)

	$(-1,0)$	$(0,1)$	$(1,3)$	$(3,4)$
x	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-3$	$-$	$-$	$-$	$+$
$g'(x)$	$-$	$+$	$-$	$+$

Based on the sign of $g'(x)$, we conclude that g is (strictly) increasing in the intervals $(0,1)$, $(3,4)$ and (strictly) decreasing on the intervals $(-1,0)$ and $(1,3)$

#6 Given: position function $s(t) = t^3 - 3t$

(a) Velocity = rate of change of position wrt. time $\quad v(t) = s'(t) = 3t^2 - 3 \quad (m/s)$

acceleration = rate of change of velocity wrt time $\quad a(t) = v'(t) = 6t \quad (m/s^2)$

(b) acceleration after 2s, $a(2) = 6(2) = 12 \ m/s^2$

(c) velocity is 0 $\Rightarrow v(t) = 0$, or $3t^2 - 3 = 0 \Rightarrow t^2 = \pm 1$. Considering only times >0,

we have velocity is 0 when $t = 1 s$.

Corresponding acceleration $a(1) = 6 \ m/s^2$

#7 (a) Given: $f(y) = \sqrt[3]{y^4 + 6}$, $g(x) = 10 - 4x^3 + 5x$

We have $f'(y) = \underbrace{\frac{1}{3}(y^4+6)^{-2/3} \cdot (4y^3)}_{\text{obtained b chain rule on } f(y)}$ and $g'(x) = -12x^2 + 5$.

Now, by chain rule

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) = \frac{1}{3}\left[(10 - 4x^3 + 5x)^4 + 6\right]^{-2/3} \cdot 4(10 - 4x^3 + 5x)^3 \cdot (-12x^2 + 5)$$

#8 (a) $f(t) = \sin\left(t^2 \cos(t)\right)$

We have $f(t) = (g \circ h)(t) = g(h(t))$ where $g(u) = \sin(u)$ and
$h(t) = t^2 \cos(t)$.

We have $g'(u) = \cos(u)$ and $h'(t) = -t^2 \sin(t) + 2t \cos(t)$
(applying product rule)

By the chain rule, $f'(t) = g'(h(t)) \cdot h'(t)$

$$\boxed{f'(t) = \cos\left(t^2 \cos(t)\right) \cdot \left(-t^2 \sin(t) + 2t \cos(t)\right)}$$

(b) $g(t) = \left(\frac{x^2-2}{x^2+1}\right)^2$

We have $g(t) = (f \circ h)(t) = f(h(t))$ where $f(u) = u^2$ and $h(t) = \frac{x^2-2}{x^2+1}$.

Since $f'(u) = 2u$ and $h'(t) = \frac{(x^2+1)\,2x - (x^2-2).2x}{(x^2+1)^2} = \frac{6x}{(x^2+1)^2}$,

by the chain rule, $\boxed{g'(t) = f'(h(t)) \cdot h'(t) = 2\left(\frac{x^2-2}{x^2+1}\right) \cdot \frac{6x}{(x^2+1)^2}.}$

Week 12 – Worksheet – MTH 305 (Spring 2017)

(1) Find the derivative of $f(x) = (2x^3+7)(3x^2-2x)$ in two ways: by using the Product Rule and by performing the multiplication first. Do your answers agree? $30x^4 - 16x^3 + 42x - 14$

(2) Find the derivative of the function

$$F(x) = \frac{3x^4 - 4x^3 + 3x^2 + 3\sqrt{x}}{x^3} \qquad -\frac{15}{2}x^{-7/2} - \frac{3}{x^2} + 3$$

in two ways: by using the Quotient Rule and by simplifying first. Show that your answers are equivalent. Which method do you prefer?

(3) Using the rules for computing derivatives, compute the derivative of the given function. At each step, specify the formula you applied.

(a) $f(x) = 4x - 3x^2 + 7$ $\quad 4-6x$

(b) $g(t) = \dfrac{x+6}{x+1}$ $\quad \dfrac{-5}{(x+1)^2}$

(c) $h(\mu) = (\mu+1)(\mu+2)(\mu+3)$ $\quad 3\mu^2 + 12\mu + 11$

(d) $f(p) = -\dfrac{2}{p^2} + 2^{\frac{1}{3}}$ $\quad 4/p^3$

(e) $h(t) = 3\sqrt{3t^2+2t+1}$ $\quad \dfrac{9t+3}{\sqrt{3t^2+2t+1}}$

(f) $j(x) = \sqrt[4]{t}(\sqrt[3]{t} + 2t + 1)$ $\quad (7/12)\, t^{-5/12} + \frac{5}{2} t^{1/4} + \frac{1}{4} t^{-3/4}$

(g) $g(x) = \dfrac{x^2+2x+1}{3\sqrt{x}}$ $\quad \dfrac{3x^2+2x-1}{6x\sqrt{x}}$

(h) $h(t) = \sqrt{\dfrac{2t-3}{t+2}}$ $\quad \dfrac{7}{2} \cdot \dfrac{1}{(t+2)^{3/2}\sqrt{2t-3}}$

(i) $f(x) = ax^3 + bx^2 + cx + d$ $\quad 3ax^2 + 2bx + c$

(4) Find equations of the tangent line and normal line to the curve at the given point.

(a) $y = \dfrac{2x+3}{x^2+4}$ at the point corresponding to $x = 0$. $\quad m = -\frac{1}{2}$ $\quad (x_0, y_0) = (0, \frac{3}{4})$ $\quad y - \frac{3}{4} = -\frac{1}{2}x$ (tangent line)

$y - \frac{3}{4} = 2x$ (normal line)

(b) $y = x + 2\sqrt{x}$ at the point corresponding to $x = 1$.

(5) Determine the intervals where each of the following functions is increasing, and determine the intervals where it is decreasing.

(a) $f(x) = \frac{1}{x+3}$. f is decreasing over its entire domain $(-\infty,-3) \cup (-3,\infty)$ since $f'(x) = -\frac{1}{(x+3)^2}$ is always negative.

(b) $g(x) = 3x^4 - 16x^3 + 18x^2$ within the interval $[-1, 4]$.

(7) Use the chain rule to compute $(f(g(x)))'$.

(a) $f(y) = \sqrt[3]{y^4+6}$, $g(x) = 10 - 4x^3 + 5x$

(b) $f(y) = \frac{3}{y}$, $g(x) = x^2 - 2x$ $\quad -\frac{3}{(x^2-2x)^2} \cdot (2x-2)$

(c) $f(y) = y^8 + 6y^3$, $g(x) = x^2 + 3x - \sqrt{3}$ $\quad \left[8(x^2+3x-\sqrt{3})^7 + 18(x^2+3x-\sqrt{3})^2\right] \cdot (2x+3)$

(8) Use the chain rule to compute the derivative of the given function.

(a) $f(x) = (x^2 + 2x - 3)^7$ $\quad 7(x^2+2x-3)^6(2x+2)$

(b) $g(x) = \sqrt[4]{x^2 + 4x^6}$ $\quad \frac{1}{4}(x^2+4x^6)^{-3/4} \cdot (2x+24x^5)$

(c) $h(x) = -\sqrt{\frac{3x+5}{x^2+2}}$ $\quad \frac{1}{2} \cdot \frac{(3x^2+10x-6)}{(x^2+2)^{3/2}(3x+5)^{1/2}}$

(d) $f(t) = \sin(t^2 \cos(t))$

(e) $g(t) = \left(\frac{x^2-2}{x^2+1}\right)^2$

(9) Find the equation of the tangent line to the graph of $f(x) = \left(\frac{3x}{x+4}\right)^3$ at the point $(-1, -1)$. $m=4$

$4x - y + 3 = 0$

(10) Compute the value of $(f \circ g)'(t)$ at the given value of t.

$$f(y) = y^3 - 3, \quad g(t) = \sqrt{t}, \quad t = 16.$$

$(f \circ g)'(t) = \frac{3}{2}\sqrt{t}$ and $(f \circ g)'(16) = 6$