(#1) Find the equation of the tangent line to the curve at the given point

(b) $y = x^3 - 3x + 1$ at $(2, 3)$

Let $f(x) = x^3 - 3x + 1$.

We have slope of tangent line $m = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ with $a = 2$.

Hence

$$m = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\left[(2+h)^3 - 3(2+h) + 1\right] - \left[2^3 - 3(2) + 1\right]}{h}$$

$$= \lim_{h \to 0} \frac{\left[8 + 12h + 6h^2 + h^3 - 6 - 3h + 1\right] - 3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{3} + \cancel{h}(9 + 6h + h^2) - \cancel{3}}{\cancel{h}} = \lim_{h \to 0} 9 + 6h + h^2$$

$$m = 9$$

Equation of tangent line through $\underset{x_1, y_1}{(2, 3)}$ $\quad y - y_1 = m(x - x_1)$ -or- $y - 3 = 9(x - 2)$

or $\boxed{9x - y - 15 = 0}$

(c) $y=\sqrt{x}$ at $(1,1)$

Let $f(x)=\sqrt{x}$

Slope of tangent line at $a=1$, $m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{1+h}-\sqrt{1}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{1+h}-1}{h}\right)\left(\frac{\sqrt{1+h}+1}{\sqrt{1+h}+1}\right)$$ multiplying by conjugate

$$= \lim_{h\to 0} \frac{(\sqrt{1+h})^2-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{1+h}+1)}$$

$$m = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}, \text{ or } m = \frac{1}{2}$$

Equation of tangent line at $(1,1)$ with slope $m=\frac{1}{2}$ is $y-1=\frac{1}{2}(x-1)$ or $\boxed{x-2y+1=0}$

(d) $y = \dfrac{2x+1}{x+2}$ at $(1,1)$

Let $f(x) = \dfrac{2x+1}{x+2}$

(We will use the other expression for slope m)

We have slope of tangent line $m = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x-a}$

(here a=1)

$$= \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{\left(\frac{2x+1}{x+2}\right) - \left(\frac{2+1}{1+2}\right)}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{2x+1}{x+2} - 1}{x-1}$$

$$= \lim_{x \to 1} \frac{\frac{2x+1-x-2}{x+2}}{x-1}$$

$$= \lim_{x \to 1} \frac{\cancel{(x-1)}}{(x+2)\cancel{(x-1)}}$$

$$= \lim_{x \to 1} \frac{1}{x+2} \quad \text{or} \quad m = \frac{1}{3}$$

Equation of tangent line through $(1,1)$ with slope $m = 1/3$

$$y - 1 = \frac{1}{3}(x-1)$$

or, $3y - 3 = x - 1$

or, $\boxed{x - 3y + 2 = 0}$

#2 (a) $\lim_{h \to 0} \frac{(1+h)^{10}-1}{h}$

Recall that the derivative of f at $x=a$,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Comparing with the given expression, we have

$$\left.\begin{aligned} f(a+h) &= (1+h)^{10} \text{ and} \\ f(a) &= 1 = 1^{10} \end{aligned}\right\} \quad \text{— (1)}$$

$f(x) = x^{10}$ and $a = 1$ satisfy eqn (1)

Hence,

$$\boxed{\begin{aligned} f(x) &= x^{10} \\ a &= 1 \end{aligned}}$$

Note: here is another choice of f, a which works

$f(x) = (1+x)^{10}$ and $a = 0$.

#3

(a) AROC between $x=100$ and $x=105$ $= \dfrac{C(105)-C(100)}{105-100}$

average rate of change

$$= \frac{\left(5000+10(105)+0.05(105^2)\right)-\left(5000+10(100)+0.05(100^2)\right)}{5}$$

$$= \frac{6601.25-6500}{5}$$

$$= 20.25$$

Similarly, (b) AROC b/w $x=100$ and $x=101$ $= \dfrac{C(101)-C(100)}{101-100}$

(c) Instantaneous rate of change of C at $x=100$ $= \displaystyle\lim_{x\to 100} \frac{C(x)-C(100)}{x-100}$

$$= \lim_{x\to 100} \frac{5000+10x+0.05x^2-6500}{x-100}$$

$$= \lim_{x \to 100} \frac{0.05x^2 + 10x - 1500}{x - 100}$$

$$= \lim_{x \to 100} \frac{\cancel{(x - 100)}(0.05x + 15)}{\cancel{(x - 100)}}$$

$$= \lim_{x \to 100} (0.05x + 15)$$

$$= 0.05(100) + 15$$

$\boxed{\text{Inst. RoC at } x = 100 \quad = \quad 20}$

Efficiency

traveling south – 95%

traveling east – 40%