Solutions to Selected Problems - Week 1

(1) Determine direct and recursive formulas for the sequence whose first few terms coincide with the given ones:

(c) $-\frac{21}{4}, -\frac{11}{2}, -\frac{23}{4}, -6, -\frac{25}{4}, \ldots$

rewriting these terms, we have $-\frac{21}{4}, -\frac{22}{4}, -\frac{23}{4}, -\frac{24}{4}, -\frac{25}{4}, \ldots$

we see that these are the first few terms of an arithmetic sequence with

first term, $a_1 = -\frac{21}{4}$ and common difference, $d = -\frac{1}{4}$.

Hence, (direct formula) $a_n = -\frac{21}{4} - \frac{1}{4}(n-1)$, $n \geq 1$.

(recursive formula) $a_1 = -\frac{21}{4}$, $a_{n+1} = a_n - \frac{1}{4}$, $n \geq 1$.

(f) $-13, 26, -52, 104, \ldots$

this is a geometric sequence with first term $b_1 = -13$ and common ratio $r = -2$.

Therefore, (direct formula) $b_n = -13(-2)^{n-1}$, $n \geq 1$.

(recursive formula) $b_1 = -13$, $b_{n+1} = -2b_n$, $n \geq 1$.

(2) For each of the following, determine whether or not they converge. If they converge, what is the limit? Provide some algebraic justification.

(c) $\left\{\left(1+\frac{1}{n}\right)^2\right\}_{n\in\mathbb{N}}$

(Scratch work / Informal reasoning)

let's write out the first few terms

$x_1 = \left(1+\frac{1}{1}\right)^2 = 2^2 = 4$

$x_2 = \left(1+\frac{1}{2}\right)^2 = \left(\frac{3}{2}\right)^2 = 2.25$

$\vdots$

$x_{10} = \left(1+\frac{1}{10}\right)^2 = \left(\frac{11}{10}\right)^2 = 1.21$

$\vdots$

$x_{100} = \left(1+\frac{1}{100}\right)^2 = (1.01)^2 = 1.0201$

we see that x_n is getting closer and closer to 1.

Formal answer - version 1 (using sequence properties)

$x_n = \left(1+\frac{1}{n}\right)^2 = 1 + \frac{2}{n} + \frac{1}{n^2}$ (using $(a+b)^2 = a^2+2ab+b^2$)

Hence, $\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)$

$= \lim_{n\to\infty} 1 + 2\lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{1}{n^2}$ (using $\lim_{n\to\infty}\frac{1}{n^\alpha} = 0$ when $\alpha \geq 1$.)

$= 1 + 2(0) + 0$

$= 1$

Therefore, the sequence converges with $L = \lim_{n\to\infty} x_n = 1$.

Formal answer - version 2

As $n\to\infty$, $\frac{1}{n}\to 0$; therefore $x_n \to 1$ as $n\to\infty$.

Hence, the sequence converges with $L = \lim_{n\to\infty} x_n = 1$.

(d) $\{(-1)^n n\}_{n\in\mathbb{N}}$

(Scratch work | informal reasoning)

evaluating a few terms,

$x_1 = (-1)^1 1 = -1$

$x_2 = (-1)^2 2 = 2$

$\vdots$

$x_{10} = (-1)^{10} 10 = 10$

$x_{11} = (-1)^{11} 11 = -11$

$\vdots$

$x_{1000} = (-1)^{1000} 1000 = 1000$

$x_{1001} = (-1)^{1001} 1001 = -1001$

We see that $|x_n|$ increases as n increases.

We have, $x_n = \begin{cases} n, & \text{if } n \text{ is even} \\ -n, & \text{otherwise.} \end{cases}$

This sequence diverges.

We will use the definition of convergence to show that there is no real number L with $L = \lim_{n\to\infty} x_n$.

Recall: A sequence $\{x_n\}_n$ is said to converge to a real number L if any open interval centered at L contains all but finitely many terms of the sequence.

Suppose, to the contrary, that such an L exists. Consider

Case ① $L=0$

Consider the interval $(-1,1)$. $x_n \notin (-1,1)$ for all $n\geq 1$. Since there are infinitely many terms of the sequence not contained in this interval, the sequence cannot converge to $L=0$.

Case ② $L>0$.

this is an open interval centered at L

Consider the interval $(0,2L)$. We see that $x_n \notin (0,2L)$ for n odd. Again, there are infinitely many terms of the sequence not contained in this interval. Hence, the sequence cannot converge to any positive L.

Case ③ $L<0$.

this is an open interval centered at L

Consider the interval $(2L,0)$. We see that $x_n \notin (2L,0)$ for n even. Again, there are infinitely many terms of the sequence not contained in this interval. Hence, the sequence cannot converge to any $L<0$.

Since there is no real L with $L = \lim_{n\to\infty} x_n$, we conclude that the sequence diverges.

(g) $\left\{3 + \frac{(-1)^n 2}{n}\right\}_{n\in\mathbb{N}}$

using limit properties, we have

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 3 + \frac{(-1)^n 2}{n} = \lim_{n\to\infty} 3 + 2\lim_{n\to\infty} \frac{(-1)^n}{n}$$

$$= 3 + 2\left(\lim_{n\to\infty} \frac{(-1)^n}{n}\right).$$

Since $-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$, we may apply the sandwich theorem to the second term.

we get $\lim_{n\to\infty} \frac{(-1)^n}{n} = \lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$

It follows that $L = \lim_{n\to\infty} x_n = 3 + 2(0) = 3.$

Hence, the sequence converges with limit $L = 3$.

(b) $\left\{ \sin\left(\frac{n\pi}{4}\right)\right\}_{n\in\mathbb{N}}$ note: first few terms are $\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, \ldots\ldots$

this sequence diverges. We show that there is no L such that $L = \lim_{n\to\infty} x_n$.

Case ① $(L=0)$ Consider the interval $(-\frac{1}{2}, \frac{1}{2})$. We see that $x_n \notin (-\frac{1}{2}, \frac{1}{2})$ for $n = 1,2,3,5,6,7,9,\ldots$

case ② $(L>0)$ — — $(0, 2L)$. We see that $x_n \notin (0, 2L)$ for $n = 4,5,6,7,8,12,13,\ldots\ldots$

case ③ $(L<0)$ — — $(2L, 0)$. We see that $x_n \notin (2L, 0)$ for $n = 1,2,3,4,8,9,10,11,12,\ldots\ldots$

In all cases, there are an infinite number of terms of the sequence which are not contained in the respective intervals.

Hence the sequence does not converge to any (real) L.

(3) Determine if the following sequences are bounded. Briefly justify your answers.

(c) $\left\{\frac{n}{n+3}\right\}_{n\in\mathbb{N}}$.

We have $x_n = \frac{n}{n+3}$

$$\leq \frac{n}{n} = 1 = M \text{ for all } n \geq 1.$$

Hence $|x_n| \leq M$ for all $n \geq 1$, where $M = 1$ is a real, positive number.

Therefore, by definition, this is a bounded sequence.

(b) $\left\{ 1 + (-1)^n (2n-1) \right\}_{n \in \mathbb{N}}$

We note that $x_n = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ 2-2n, & \text{if } n \text{ is odd.} \end{cases}$

as n increases, x_n increases without bound

this is an unbounded sequence.

(Formal justification)

Suppose the sequence is bounded with bound M. this is a positive real number

Choose an even, positive integer K such that $K > M$.

Then $|x_K| = 2K > M$,

which contradicts the assumption that the sequence is bounded.

Scratch work

(n even) $1 + (-1)^n (2n-1)$

$= 1 + (2n-1)$ $(-1)^n = 1$ for n even

$= 2n$

(n odd)

$1 + (-1)^n (2n-1) = 1 - (2n-1)$

$= 2 - 2n$

Since $(-1)^n = -1$ for n odd

Note: problems with terms of the form a^n (ex: $\left(\frac{1}{2}\right)^n$, 4^n, $2+\left(\frac{-1}{6}\right)^n$, ...)

let's consider $2+\left(-\frac{1}{6}\right)^n$

evaluating a few terms

$$x_1 = 2+\left(-\frac{1}{6}\right)^1 = 2-\frac{1}{6} = \frac{11}{6} = 1.8333\ldots$$

$$x_2 = 2+\left(\frac{-1}{6}\right)^2 = 2+\frac{1}{36} = \frac{73}{36} = 2.02777\ldots\ldots$$

$$x_{10} = 2+\left(\frac{-1}{6}\right)^{10} = 2.00000001654$$

$$\vdots$$

we see that as $n\to\infty$, $x_n \to 2$.

therefore the series is convergent with $L=2$.

alternatively, using sequence properties

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} 2 + \lim_{n\to\infty}\left(-\frac{1}{6}\right)^n = 2+\lim_{n\to\infty}\left(-\frac{1}{6}\right)^n = 2+0 = 2.$$

The last equality is obtained by applying the sandwich theorem to the 2^{nd} term.

Since $-\frac{1}{n} \leq \left(\frac{-1}{6}\right)^n \leq \frac{1}{n}$, we have $\lim_{n\to\infty}\left(-\frac{1}{6}\right)^n = \lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty}\frac{1}{n} = 0$.

In general, you will observe that as $n\to\infty$, $a^n \to 0$ only if $|a|<1$