FINAL REVIEW

(I) #2 $f(x) = x + \frac{1}{x^2}$ with $F(x) = \frac{1}{2}$ at $x=1$ and $F(x) = \frac{3}{2}$ at $x=-1$.

Note that domain of f is $(-\infty, 0) \cup (0, \infty)$

Hence, the antiderivatives of f take the form

$$F(x) = \begin{cases} \frac{x^2}{2} - \frac{1}{x} + C_1 & x<0 \\ \frac{x^2}{2} - \frac{1}{x} + C_2 & x>0 \end{cases}$$

Note: each interval has a unique const. of integration.

Since $F(x) = \frac{1}{2}$ at $x=1$, $\frac{1}{2} - 1 + C_2 = \frac{1}{2} \Rightarrow C_2 = 1$

$F(x) = \frac{3}{2}$ at $x=-1$, $\frac{1}{2} + 1 + C_1 = \frac{3}{2} \Rightarrow C_1 = 0$

Therefore,

$$\boxed{F(x) = \begin{cases} x^2/2 - 1/x & \text{if } x<0 \\ x^2/2 - 1/x + 1 & \text{if } x>0 \end{cases}}$$

#4 (1) $\displaystyle\int_{-1}^{1} |2x+1|\,dx$

We have $|2x+1| = \begin{cases} -(2x+1), & x < -\frac{1}{2} \\ 2x+1 & x \geq -\frac{1}{2} \end{cases}$

Hence, $\displaystyle\int_{-1}^{1} |2x+1|\,dx = \int_{-1}^{-1/2} -(2x+1)\,dx + \int_{-1/2}^{1} (2x+1)\,dx$

$$= -2\int_{-1}^{-1/2} x\,dx - \int_{-1}^{-1/2} dx + 2\int_{-1/2}^{1} x\,dx + \int_{-1/2}^{1} dx$$

$$= -2\left(\frac{x^2}{2}\Big|_{-1}^{-1/2}\right) - \left(x\Big|_{-1}^{-1/2}\right) + 2\left(\frac{x^2}{2}\Big|_{-1/2}^{1}\right) + \left(x\Big|_{-1/2}^{1}\right)$$

$$= -1\left[(-\tfrac{1}{2})^2 - (-1)^2\right] - \left[-\tfrac{1}{2} - (-1)\right] + \left[1 - (-\tfrac{1}{2})^2\right] + \left[1 - (-\tfrac{1}{2})\right]$$

$$= \frac{3}{4} - \frac{1}{2} + \frac{3}{4} + \frac{3}{2}$$

$$= \boxed{\frac{5}{2}}$$

#5

Given: Initial height, $h = 30$ ft

time to hit ground, $T = 1$ sec

(time to travel 30ft down)

Also know: velocity changes at a $\boxed{\text{constant}}$ rate

a = rate of acceleration due to gravity $= -32\ ft/sec^2$

negative since we travel down from roof

initial velocity

Hence $v(t) = v_0 + at$

Distance covered $h = \int v(t)\,dt$

$$\Rightarrow -30 = \int_0^1 (v_0 + at)\,dt$$

since ball is dropped from roof

$$\Rightarrow -30 = \int_0^1 (v_0 - 32t)\,dt = v_0 \left(t\Big|_0^1\right) - 32\left(\frac{t^2}{2}\Big|_0^1\right)$$

$$\Rightarrow -30 = v_0 - 16, \text{ or } \boxed{v_0 = -14\ ft/sec}$$

velocity at impact $v_f = v(1) = v_0 - 32(1) = -14 - 32 = \boxed{-46\ ft/sec}$

#6 (ii) $\int \sin(x)\sin(\cos(x))\,dx$

Let $u(x) = \cos(x)$

$\Rightarrow \quad u'(x) = \frac{du}{dx} = -\sin(x)$

$\Rightarrow \quad -du = \sin(x)\,dx$

$$\int \sin(x)\sin(\cos(x))\,dx = -\int \sin(u)\,du$$

$$= -\left[-\cos(u) + c\right]$$

$$= \boxed{\cos(\cos(x)) + C}$$

Note:

$\frac{d}{du}\left[\cos(u)\right] = -\sin(u)$, or

$\frac{d}{du}\left[-\cos(u)\right] = \sin(u)$

Hence the antiderivative of $\sin(u)$ is $-\cos(u)$

(I) #1 (iii) $h(x) = \sqrt[3]{\dfrac{x^2-x}{x^2}}$

Let $f(u) = \sqrt[3]{u} = u^{1/3}$ and

$g(x) = \dfrac{x^2-x}{x^2} = 1 - \dfrac{1}{x}$

We have $f'(u) = \dfrac{1}{3}u^{-2/3}$

$g'(x) = \dfrac{1}{x^2}$

Then $h(x) = (f \circ g)(x) = f(g(x))$

$\Rightarrow h'(x) = f'(g(x)) \cdot g'(x)$ (by chain rule)

$$= \boxed{\frac{1}{3}\left(\frac{x^2-x}{x^2}\right)^{-2/3} \cdot \left(\frac{1}{x^2}\right)}$$

#3 Given: $(x_0, y_0) = (-1, -1)$

Slope of tangent line, $m = f'(x)$

$m = 3\left(\dfrac{3x}{x+4}\right)^2 \cdot \left[\dfrac{3(x+4) - 3x}{(x+4)^2}\right]$ (by chain rule)

$m = \dfrac{36(3x)^2}{(x+4)^4}$

Slope at $x=-1$, $m = \dfrac{36 \cdot 3^2}{3^4} = 4$

Eqn. of tangent line

$y - y_0 = m(x - x_0)$, or $y - (-1) = 4(x - (-1))$

$y + 1 = 4(x+1)$

or $\boxed{y = 4x + 3}$

(III) #2 (i) $f(x) = \begin{cases} 2x+3 & \text{if } x<0 \\ x^2+3 & \text{if } x \geq 0 \end{cases}$ $\qquad g(x) = \dfrac{x+1}{x+2}$

Note: $\mathrm{dom}(f) = \mathbb{R}$ $\qquad \mathrm{dom}(g) = \mathbb{R}\setminus\{-2\}$

$$(f\circ g)(x) = f(g(x)) = \begin{cases} 2\left(\dfrac{x+1}{x+2}\right)+3 & \text{if } \left(\dfrac{x+1}{x+2}\right) < 0 \\ \left(\dfrac{x+1}{x+2}\right)^2+3 & \text{if } \left(\dfrac{x+1}{x+2}\right) \geq 0 \end{cases}$$

$$= \begin{cases} \dfrac{5x+8}{x+2} & \text{if } x \in (-2,-1) \\ \dfrac{4x^2+14x+13}{x^2+4x+4} & \text{if } x \in (-\infty,-2) \cup [-1,\infty) \end{cases}$$

note: $\dfrac{x+1}{x+2} < 0$ when

- $(x+1) < 0$ and $(x+2) > 0$

or

- $(x+1) > 0$ and $(x+2) < 0$

Equivalently, when

- $x < -1$ and $x > -2$, i.e., $x \in (-2,-1)$
- $x > -1$ and $x < -2$, no such x exists

$\mathrm{dom}(f\circ g) = \mathbb{R}\setminus\{-2\}$ $\quad (\mathrm{dom}(g) = \mathbb{R}\setminus\{-2\},\ \mathrm{dom}(g) \subset \mathrm{dom}(f))$

$$(g\circ f)(x) = g(f(x)) = \begin{cases} \dfrac{(2x+3)+1}{(2x+3)+2} & \text{if } x<0 \\ \dfrac{(x^2+3)+1}{(x^2+3)+2} & \text{if } x \leq 0 \end{cases} = \begin{cases} \dfrac{2x+4}{2x+5} & \text{if } x<0 \\ \dfrac{x^2+4}{x^2+5} & \text{if } x \leq 0 \end{cases}$$

$\mathrm{dom}(g\circ f) = \mathbb{R}\setminus\{-\tfrac{5}{2}\}$ $\quad (\mathrm{dom}(f) = \mathbb{R}$, want $f(x) \neq -2$, or $2x+3 \neq -2$, or $x \neq -5/2)$

#3 (ii) $\lim\limits_{x\to 3} \dfrac{x^2+4x-21}{x-3}$

$= \lim\limits_{x\to 3} \dfrac{\cancel{(x-3)}(x+7)}{\cancel{(x-3)}}$

$= \lim\limits_{x\to 3} (x+7)$

$= 3+7$

$= \boxed{10}$

(iv) #2 (ii) $\left\{\sqrt{n^2+4}-n\right\}_{n\in\mathbb{N}}$

$x_n = \sqrt{n^2+4}-n$

$= \left(\sqrt{n^2+4}-n\right) \cdot \dfrac{\left(\sqrt{n^2+4}+n\right)}{\left(\sqrt{n^2+4}+n\right)}$

$= \dfrac{\left(\sqrt{n^2+4}\right)^2-n^2}{\sqrt{n^2+4}+n}$

$= \dfrac{(\cancel{n^2}+4)-\cancel{n^2}}{\sqrt{n^2+4}+n}$

$= \dfrac{4}{\sqrt{n^2+4}+n}$

As n increases, we see that x_n gets smaller.

We can make x_n arbitrarily small by choosing n sufficiently large

Hence $\left\{\sqrt{n^2+4}-n\right\}_{n\in\mathbb{N}}$ converges with

$\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}\left(\sqrt{n^2+4}-n\right) = 0.$

(IV) #4 (ii) Compute the Sum $S = 6561 - 2187 + 729 + \cdots - 3$

The terms of this series form a geometric sequence with

first term $a = 6{,}561$ and

common ratio $r = \frac{-2{,}187}{6{,}561} = -\frac{1}{3}$

We now find the number of terms, n

Since $a_n = -3$, we have $ar^{n-1} = -3$, or

$$6561 \left(-\tfrac{1}{3}\right)^{n-1} = -3$$

$$\Rightarrow 6{,}561 \left(-\tfrac{1}{3}\right)^{n-1} = -3$$

$$\Rightarrow \left(-\tfrac{1}{3}\right)^{n-1} = \frac{-1}{2{,}187} = \frac{-1}{3^7} = \left(-\tfrac{1}{3}\right)^7$$

$$\Rightarrow n-1 = 7 \text{ or } \underline{n = 8}$$

Now, $S = \frac{a(r^n - 1)}{r-1} = \frac{6{,}561\left[\left(-\frac{1}{3}\right)^8 - 1\right]}{-\frac{1}{3} - 1} = -\frac{6{,}561}{4/3}\left[\frac{1}{6{,}561} - 1\right] = \frac{6560}{4/3} = \boxed{4920}$

(Sum to n terms)

Final Exam Review – MTH 305 (Spring 2017)

(I) Integration

1. Find the antiderivatives of $f(x) = |4x - 1|$.

$$F(x) = \begin{cases} -2x^2 + x + C & \text{if } x \le 1/4 \\ 2x^2 - x + \frac{1}{4} + C & \text{if } x > 1/4 \end{cases}$$

2. Find the antiderivative of $f(x) = x + \frac{1}{x^3}$ that takes the value $1/2$ at $x = 1$ and $3/2$ at $x = -1$.

3. In the following exercises, find the domain of the integrand, then evaluate the indefinite integral.

   (i) $\displaystyle\int \left(e^x - \frac{1}{\sqrt[3]{x}}\right) dx$ $\quad e^x - \dfrac{3x}{2\sqrt[3]{x}} + C$

   (ii) $\displaystyle\int \left(-x^{2/3} + x\right)\left(\frac{1}{x} - 3\right) dx$ $\quad \dfrac{9x^{5/3}}{5} - \dfrac{3x^{2/3}}{2} - \dfrac{3x^2}{2} + x + C$

4. In the following exercises, evaluate the definite integral.

   (i) $\displaystyle\int_{-1}^{1} |2x + 1|\, dx$

   (ii) $\displaystyle\int_{1}^{2} \frac{2x^2 + 9x + 8}{x^5}\, dx$ $\quad \dfrac{21}{4}$

5. A ball is dropped with a downward initial velocity from a roof 30 feet high. The ball hits the ground after 1 second. What is the initial velocity of the ball? What is the velocity of the ball at impact?

6. In the following exercises, evaluate the given integral. Use differentiation to justify your answers.

   (i) $\displaystyle\int x(x^2 + 3)\sqrt{x^2 + 3}\, dx$ $\quad \dfrac{1}{5}(x^2+3)^{5/2} + C$

   (ii) $\displaystyle\int \sin(x)\sin(\cos(x))\, dx$

   (iii) $\displaystyle\int_{0}^{1} \frac{x^3}{\sqrt{(x^4 + 1)^3}}\, dx$ $\quad \dfrac{1}{4}(2 - \sqrt{2})$

7. Others: See HW6 Q1, HW6 Q6

(II) Differentiation

1. In the following exercises, compute the derivative of the given function.

   (i) $f(x) = \dfrac{\sqrt{x} + 2}{\sqrt{x} - 2}$ $\quad \dfrac{-2}{\sqrt{x}(\sqrt{x}-2)^2}$

   (ii) $g(x) = (4 - x^2)(2x + 7)$ $\quad -6x^2 - 14x + 8$

   (iii) $h(x) = \sqrt[3]{\dfrac{x^2 - x}{x^2}}$

2. Compute the value of $(f \circ g)'(t)$ at $t = 2$ if $f(y) = \frac{y-2}{y}$ and $g(t) = t^4 - t^2$. $\quad \dfrac{7}{18}$

3. Find the equation of the tangent line to the graph of $f(x) = \left(\dfrac{3x}{x + 4}\right)^3$ at the point $(-1, -1)$.

4. The velocity of an ice skater moving along a horizontal line is $v(t) = 2(t - 3)^2 - 2$ feet per second, where t is the time in seconds.

   initial speed = $|v(0)|$ = 16 ft/s

   (i) Plot the velocity of the skater for t between 0 and 8 seconds, and determine his initial speed.

(ii) Determine the acceleration of the skater as a function of time, and calculate his initial acceleration. $a(t) = 4t - 12$ $a(0) = -12$ ft/s^2

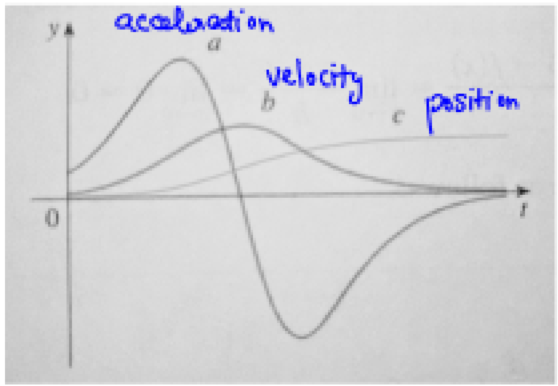
(iii) When does the skater reverse his motion? Based on these results, describe the skater's motion.

at $t=2$, $t=4$ s
$v(t)$ changes sign

(iv) At what moment is the skater's velocity minimal, and what is the velocity then?

$t=3$, $v(3) = -2$ ft/sec

5. The figure below shows the graph of three functions. One is the position function of a car, one is the velocity of the car, and one is the acceleration. Identify each curve and explain your choices.



c is increasing and b is positive
b is derivative of c

When b is increasing, a is positive, when b is decreasing a is negative
a is derivative of b

6. Others: see Week 10 Worksheet Q3 Q5 and Quiz 4 Q3

(III) Functions

1. Determine if the given equation describes y as a function of x.

(i) $y = -7$ YES (ii) $x = 7y^2 + 5$ NO (iii) $y^2 = 2x^2 - 1$ NO (iv) $y = \sqrt{x - 12}$ YES (v) $7y^3 + 2x = -12$ YES

$$(vi)\ y = \begin{cases} x + 2 & \text{if } x < 0, \\ 1 - x & \text{if } x > 0. \end{cases}$$ YES

see exam 1 review #10

2. Determine the assignment rule and the domain of the specified composite functions.

(i) $f \circ g$ if $f(x) = x - 3$ and $g(x) = \sqrt{x}$. $\sqrt{x} - 3$, $x \in [0, \infty)$

(ii) $f \circ g$ and $g \circ f$ if $f(x) = \begin{cases} 2x + 3 & \text{if } x < 0, \\ x^2 + 3 & \text{if } x \geq 0 \end{cases}$ and $g(x) = \dfrac{x+1}{x+2}$.

3. Evaluate the limit, if it exists

(i) $\displaystyle\lim_{x \to -3} \frac{\sqrt{x^2 + 16} - 5}{x + 3}$ $-6/10$ see exam 1 review #16

(ii) $\displaystyle\lim_{x \to 3} \frac{x^2 + 4x - 21}{x - 3}$

4. Find the values of A and B so that the following function is continuous for all values of x

$$f(x) = \begin{cases} Ax - B & \text{if } x \leq -1 \\ 2x^2 + 3Ax + B & \text{if } -1 < x \leq 1 \\ 4 & \text{if } x > 1 \end{cases}$$

$A = 3/4$ $B = -1/4$
see week 8 #5

5. Others: Midterm Exam review Q13 Q15, Week 8 Worksheet Q1 Q2

(IV) Sequences/Series

1. Determine direct and recursive formulas for the given sequences below. Give the value of the sequence's first term.

   (i) An arithmetic sequence $\{x_n\}_{n\geq 1}$ with $x_3 = 2$ and a common difference of $\frac{1}{4}$.

   $x_{n+1} = x_n + \frac{1}{4},\ n \geq 1$
   $x_1 = \frac{3}{2}$
   $x_n = \frac{3}{2} + (n-1)\frac{1}{4},\ n \geq 1$

   (ii) An geometric sequence $\{x_n\}_{n\geq 1}$ with $x_5 = 64$ and $x_2 = 512$.

   $x_{n+1} = \frac{x_n}{2},\ n \geq 1$
   $x_1 = 1024$
   $x_n = 1024\left(\frac{1}{2}\right)^{n-1}$

2. For each of the following, determine whether or not they converge. If they converge, what is the limit? Provide some algebraic justification.

   (i) $\left\{\sin\left(\frac{n\pi}{2}\right)\right\}_{n\in\mathbb{N}}$ NO

   (ii) $\left\{\sqrt{n^2+4} - n\right\}_{n\in\mathbb{N}}$

   (iii) $\left\{-2 + \frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ Yes, limit is -2

3. Decide whether the geometric series converge or diverge. Justify your answer. If the series converges, compute its sum.

   (i) $5 - 5 + 5 - 5 + 5 - 5 \cdots +$ No

   (ii) $\sum_{n=1}^{\infty} \frac{4}{7^{n-1}}$ $\frac{98}{3}$

4. Compute the sum without using a calculator.

   (i) $S = 2 + 8 + 14 + \cdots + 638$ $34,240 = (107)(320)$

   (ii) $S = 6561 - 2187 + 729 + \cdots - 3$

5. Others: Exam 1 review Q5 Q6, Midterm Exam Q11