Selected Solutions / Solution sketch (Midterm exam review)

#3 (c) An arithmetic sequence $\{x_n\}_{n\geq 1}$ with $x_8 = 40$ and $x_{20} = -20$.

We have, $x_n = x_1 + (n-1)d, \quad n \geq 1$ (n^{th} term of an arithmetic sequence)

for $n=8$, we have $40 = x_1 + (8-1)d$ -or- $40 = x_1 + 7d$ —— (i)

$n=20$, " — $-20 = x_1 + (20-1)d$ $\quad -20 = x_1 + 19d$ —— (ii)

Solving equations (i) and (ii) for x_1 and d, we get

(i) − (ii) gives $60 = -12d$ or $\underline{d = -5}$

substituting this in (i) gives $\underline{x_1 = 75}$

Therefore,

(Direct formula) $x_n = 75 - 5(n-1), \quad n \geq 1$

(Recursive formula) $x_{n+1} = x_n - 5, \quad n \geq 1$

$x_1 = 75$

(#3) (d) A geometric sequence $\{x_n\}_{n\geq 1}$ with $x_5 = 64$ and $x_2 = 512$.

We have, $x_n = x_1 r^{n-1}$, $n \geq 1$ (n^{th} term of a geometric sequence)

for $n=5$, we have $64 = x_1 r^{5-1}$ — or — $64 = x_1 r^4$ —— (i)

$n=2$, — " — $512 = x_1 r^{2-1}$ — or — $512 = x_1 r$ —— (ii)

Solving equations (i) and (ii) for x_1 and r, we get

dividing (i) by (ii), $\frac{64}{512} = \frac{r^4}{r} = r^3$

or, $\frac{1}{8} = r^3 \Rightarrow r = \frac{1}{2}$

Substituting in (ii) gives $x_1 = \frac{512}{(1/2)} = 1024$.

Therefore,

$$\boxed{\begin{array}{ll} \text{(direct formula)} & x_n = 1024\left(\frac{1}{2}\right)^{n-1}, \; n \geq 1 \\ \text{(recursive formula)} & x_{n+1} = x_n/2, \; n \geq 1 \\ & x_1 = 1024 \end{array}}$$

#4 (c) $\{\sqrt{n^2+3}-n\}_{n\in\mathbb{N}}$

Let $x_n = \sqrt{n^2+3} - n$

(Multiplying and dividing by $(\sqrt{n^2+3}+n)$, we get)

$$= \frac{(\sqrt{n^2+3}-n)}{1} \cdot \frac{(\sqrt{n^2+3}+n)}{(\sqrt{n^2+3}+n)}$$

$$= \frac{(\sqrt{n^2+3})^2 - n^2}{\sqrt{n^2+3}+n}$$

$$= \frac{3}{\sqrt{n^2+3}+n}$$

as n increases, we see that x_n gets smaller.

We can make x_n infinitesimally small by choosing n sufficiently large.

Hence,

$$\boxed{\begin{array}{c}\{(\sqrt{n^2+3}-n)\}_{n\in\mathbb{N}} \text{ converges with} \\ \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}(\sqrt{n^2+3}-n) = 0.\end{array}}$$

Note: we have $x_n = \sqrt{n^2+3} - n = \dfrac{3}{\sqrt{n^2+3}+n}$

applying limit laws/ properties, we get

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(\frac{3}{\sqrt{n^2+3}+n}\right) = \lim_{n\to\infty} \frac{\cancel{n}\cdot 3/n}{\cancel{n}\left(\sqrt{1+3/n^2}+1\right)}$$

$$= \frac{\lim_{n\to\infty} 3/n}{\sqrt{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 3/n^2} + \lim_{n\to\infty} 1}$$

$$= \frac{0}{\sqrt{1+0}+1}$$

$$= \frac{0}{2} = 0.$$

Hence $\boxed{\lim_{n\to\infty} \left(\sqrt{n^2+3} - n\right) = 0.}$

#4 (a) $\left\{ \cos\left(\frac{n\pi}{2}\right) \right\}_{n \in \mathbb{N}}$

Let $x_n = \cos\left(\frac{n\pi}{2}\right)$, $n \in \mathbb{N}$

We have

(for $n=1$) $x_1 = \cos\left(\frac{\pi}{2}\right) = 0$

(for $n=2$) $x_2 = \cos(\pi) = -1$

$\vdots$

(for $n=4$) $x_4 = \cos(2\pi) = 1$

$\vdots$

x_n does not approach any single real number L as n increases

Hence $\boxed{\left\{ \cos\left(\frac{n\pi}{2}\right) \right\}_{n \in \mathbb{N}} \text{ does not converge.}}$

#4 (b) $\left\{ (-1)^n n \right\}_{n \in \mathbb{N}}$

We have

($n=1$) $x_1 = (-1) \cdot 1 = -1$

($n=2$) $x_2 = (-1)^2 \cdot 2 = 2$

$\vdots$

($n=10$) $x_{10} = (-1)^{10} \cdot 10 = 10$

$\vdots$

($n=999$) $x_{999} = (-1)^{999} \cdot 999 = -999$

x_n does not approach any single real number L as n increases

Hence $\boxed{\left\{ (-1)^n n \right\}_{n \in \mathbb{N}} \text{ does not converge.}}$

#5 (b) $\left\{ \frac{n}{n+5} \right\}_{n \in \mathbb{N}}$

Let $x_n = \frac{n}{n+5}$, $n \in \mathbb{N}$

Since $\frac{n}{n+5} < \frac{n}{n} = 1$ for all $n \in \mathbb{N}$,

we have $|x_n| \le 1$ for all $n \in \mathbb{N}$

By definition, $\boxed{\left\{ \frac{n}{n+5} \right\}_{n \in \mathbb{N}} \text{ is a bounded sequence}}$

#6 (b) $\left\{ \frac{5n}{n+3} \right\}_{n \in \mathbb{N}}$

Let $x_n = \frac{5n}{n+3}$, $n \in \mathbb{N}$

we have $x_n = \frac{\cancel{n} \cdot 5}{\cancel{n}(1+3/n)} = \frac{5}{1+3/n}$

Now applying limit properties,

$$\lim_{n\to\infty} x_n = \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} \left(1 + 3/n\right)}$$ (property (9)(e))

$$= \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} 3/n}$$ (by (9)(a))

$$= \frac{\lim_{n\to\infty} 5}{\lim_{n\to\infty} 1 + 3 \cdot \lim_{n\to\infty} \frac{1}{n}}$$ (by (9)(c))

$$= \frac{5}{1 + 3\cdot 0}$$ (by (9)(g) and $\lim_{n\to\infty} \frac{1}{n} = 0$)

$$\boxed{\lim_{n\to\infty} x_n = 5.}$$

#7 (c) $\sum_{n=1}^{\infty} \frac{7}{8^{n-1}}$

$$\sum_{n=1}^{\infty} \frac{7}{8^{n-1}} = 7\sum_{n=1}^{\infty} \frac{1}{8^{n-1}} = 7\sum_{n=1}^{\infty} \frac{1}{8^n/8} = 56\sum_{n=1}^{\infty} \frac{1}{8^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{8^n} = \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \dots\dots$

is a geometric series with $a = \frac{1}{8}$ and $r = \frac{1}{8}$,

it's sum $L = \frac{a}{1-r} = \frac{1/8}{1-1/8} = \frac{1}{7}$

Hence, $\sum_{n=1}^{\infty} \frac{7}{8^{n-1}} = 56\left(\frac{1}{7}\right) = 8.$

#9 (a) $S = 6 - 3 + \frac{3}{2} - \frac{3}{4} + \cdots - \frac{3}{64}$

the terms of the sum S form a geometric sequence with $a = 6$ and $r = -\frac{1}{2}$.

We have $a_n = a r^{n-1}$, $n \geq 1$ (n^{th} term of a geometric sequence)

hence, $-\frac{3}{64} = 6\left(\frac{-1}{2}\right)^{n-1}$

$$\Rightarrow \frac{-3}{64} \cdot \frac{1}{6} = \left(\frac{-1}{2}\right)^{n-1}$$

$$\Rightarrow -\frac{1}{128} = \left(\frac{-1}{2}\right)^{n-1}$$

$$\Rightarrow \frac{(-1)}{2^7} = \frac{(-1)^{n-1}}{2^{n-1}}$$

which yields $\underline{n = 8}$

Now, we have

$$S = \frac{a(1-r^n)}{1-r}$$

$$= \frac{6\left(1 - \left(-\frac{1}{2}\right)^8\right)}{\left(1 - \left(-\frac{1}{2}\right)\right)}$$

$$= \frac{6\left(1 - \frac{1}{2^8}\right)}{3/2}$$

$$\boxed{S = 4\left(1 - \frac{1}{256}\right)}$$

#12 (b) $f(x) = \frac{x}{x^2+1}$ and $g(x) = \sqrt{x}$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{x^2+1}\right) = \sqrt{\frac{x}{x^2+1}}$$

Domain of $g \circ f$

- want $x \in$ domain(f)

  since domain$(f) = \mathbb{R}$, any real number x works.

- we also need $f(x) \in$ domain(g)

  since domain$(g) = [0, \infty)$, we require

  $$\frac{x}{x^2+1} \geq 0$$

  this holds when $x \geq 0$.

Therefore, domain $(g \circ f) = [0, \infty)$

(or $\{x \in \mathbb{R} \mid x \geq 0\}$

#16 (a)

$$\lim_{x \to -3} \frac{\sqrt{x^2+16} - 5}{x+3}$$

Let $x_n = \dfrac{\sqrt{x^2+16} - 5}{x+3}$

$$= \frac{\sqrt{x^2+16} - 5}{x+3} \cdot \frac{(\sqrt{x^2+16} + 5)}{(\sqrt{x^2+16} + 5)}$$

$$= \frac{(\sqrt{x^2+16})^2 - 5^2}{(x+3)(\sqrt{x^2+16}+5)}$$

$$= \frac{x^2 + \overbrace{16 - 25}^{-9}}{(x+3)(\sqrt{x^2+16} + 5)}$$

$$= \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(\sqrt{x^2+16} + 5)}$$

Hence, $\displaystyle\lim_{x \to -3} x_n = \lim_{x \to -3} \frac{x-3}{\sqrt{x^2+16} + 5} = \frac{\lim_{x \to -3} (x-3)}{\lim_{x \to -3} (\sqrt{x^2+16} + 5)} = \frac{(-3)-3}{\sqrt{(-3)^2+16} + 5} = \frac{-6}{5+5} = \frac{-6}{10}$

Other problems - Solutions/Sketch

(1) Determine direct and recursive formulas for the sequences whose first few terms coincide with the given ones:

(a) $6, -12, -30, -48, \ldots$

Arithmetic Seq. $x_1 = 6$, $d = -18$ (Direct) $x_n = 6 - 18(n-1)$, $n \geq 1$
(Recursive) $x_{n+1} = x_n - 18$, $n \geq 1$ with $x_1 = 6$

(b) $2, -4, 8, -16, \ldots$

Geometric Seq. $a_1 = 2$; $r = -2$ (Direct) $a_n = 2(-2)^{n-1}$, $n \geq 1$
(Recursive $a_{n+1} = -2a_n$, $n \geq 1$
$a_1 = 2$

(2) Determine a recursive formula for the sequence whose first few terms coincide with the given ones:

(a) $3, 6, 9, 15, 24, 39, 63 \ldots$

$a_{n+1} = a_n + a_{n-1}$, $n \geq 2$ with $a_1 = 3$ and $a_2 = 6$.

(b) $6, 18, 54, 162, \ldots$

$a_{n+1} = 3a_n$, $n \geq 1$ with $a_1 = 6$.

(3) Determine direct and recursive formulas for the given sequences below. Give the value of the sequence's first term.

(a) An arithmetic sequence $\{x_n\}_{n\geq 1}$ with $x_3 = 4$ and a common difference of $\frac{1}{3}$.

$x_n = \frac{10}{3} + \frac{(n-1)}{3}$ $n \geq 1$
$x_{n+1} = x_n + \frac{1}{3}$ $x_1 = 10/3$

(b) A geometric sequence $\{x_n\}_{n\geq 1}$ with $x_2 = 4$ and a common ratio of $-\frac{1}{4}$.

$x_n = -16\left(-\frac{1}{4}\right)^{n-1}$ $n \geq 1$; $x_1 = -16$
$x_{n+1} = -x_n/4$

(4) For each of the following, determine whether or not they converge. If they converge, what is the limit? Provide some algebraic justification.

(a) $\left\{\cos\left(\frac{n\pi}{2}\right)\right\}_{n\in\mathbb{N}}$

(b) $\{(-1)^n n\}_{n\in\mathbb{N}}$

(c) $\left\{\sqrt{n^2+3}-n\right\}_{n\in\mathbb{N}}$

See previous pages

(d) $\left\{-1+\frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ $\rightarrow$ converges to $L=-1$

(intuition) as $n \to \infty$ $\frac{(-1)^n}{n} \to 0$

(5) Determine if the following sequences are bounded. Briefly justify your answers.

(a) $\{(-1)^n(2n+1)-3\}_{n\in\mathbb{N}}$ $\rightarrow$ not bounded

(b) $\left\{\frac{n}{n+5}\right\}_{n\in\mathbb{N}}$ $\rightarrow$ (bounded) see previous pages for justification

(6) For each of the following, determine whether or not the given sequence is convergent or divergent. If convergent, determine its limit. Use the properties of limits discussed in class to justify your answer.

(a) $\{(-1)^n+3\}_{n\in\mathbb{N}}$ $\rightarrow$ divergent

(b) $\left\{\frac{5n}{n+3}\right\}_{n\in\mathbb{N}}$ $\rightarrow$ See previous page for work

(c) $\left\{\frac{2-(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ $\rightarrow$ Convergent, $L=0$

(main steps)

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2}{n} - \lim_{n\to\infty} \frac{(-1)^n}{n}$$

$\lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

can apply sandwich theorem here

$$-\frac{1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n}$$

(7) Decide whether the geometric series converge or diverge. Justify your answer. If the series converges, compute its sum.

(a) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$ Converges $a = 1$, $r = -\frac{1}{3}$ $\quad L = \frac{1}{1-(-\frac{1}{3})} = \frac{3}{4}$

(b) $3 - 3 + 3 - 3 + 3 - 3 \dots +$ diverges since $|r| = |-1| = 1$.

(c) $\sum_{n=1}^{\infty} \frac{7}{8^{n-1}}$ worked out in prior page

(d) $\frac{3}{4} + \frac{9}{16} + \dots + \left(\frac{3}{4}\right)^n + \dots$ Converges $a = \frac{3}{4}$, $r = \frac{3}{4}$ $\quad L = \frac{3/4}{1 - 3/4} = 3$

(8) (*Writing recurring decimals as fractions*) Let $x = 0.\overline{35}$. Show that $x = \frac{35}{99}$. (note: the notation $0.\overline{35}$ means that the digits 3 and 5 repeat indefinitely; i.e., $x = 0.353535353535\dots$. Hint: try and write x as a geometric series!)

$$0.\overline{35} = 0.35353535\dots = \underbrace{\frac{35}{100} + \frac{35}{10000} + \frac{35}{10^6} + \dots}_{\text{Geometric series } a = \frac{35}{100},\ r = \frac{1}{100}}$$

$$L = \frac{a}{1-r} = \frac{35/100}{1 - 1/100} = \frac{35}{99}$$

(9) Compute the sum without using a calculator.

(a) $S = 6 - 3 + \frac{3}{2} - \frac{3}{4} + \cdots - \frac{3}{64}$ $\rightsquigarrow$ worked out in previous pages

(b) $S = 3 + 8 + 13 + \cdots + 48$ $\rightsquigarrow$ arithmetic series $a = 3$ $d = 5$

(find n) $48 = 3 + 5(n-1)$ $\rightsquigarrow$ gives $n = 10$

(Sum to n terms) $S = \frac{n}{2}(a_1 + a_n) = \frac{10}{2}(3 + 48) = 5(51) = 255$

(10) Determine if the given equation describes y as a function of x.

(a) $y = -3$ Yes

(b) $x = 2y^2 + 3$ No (counterexample) $x = 5$, $y = \pm 1$ $\rightsquigarrow$ multiple y values for same x value

(c) $y^2 = 2x^2 - 1$ No (counterexample) $x = 1$, $y = \pm 1$ $\rightsquigarrow$ multiple y values for same x value

(d) $y = \sqrt{x - 4}$ Yes

(e) $2y^3 + 5x = 10$ Yes

(f) $y = \begin{cases} x + 2 & \text{if } x < 0, \\ 1 - x & \text{if } x > 0. \end{cases}$ Yes

(11) Determine the domain of the given function.

(a) $f(x) = \frac{1}{\sqrt[2]{x^2 - 8x}}$ $\{x \in \mathbb{R} \mid x < 0 \text{ or } x > 8\}$ Hint: want $x^2 - 8x > 0$ $\Rightarrow x(x-8) > 0$

(b) $g(x) = \frac{x - 3}{x^2 - 16}$ $\{x \in \mathbb{R} \mid x \neq \pm 4\}$

(c) $h(t) = \sqrt[3]{3t - 2}$ $\mathbb{R}$

(12) Determine the assignment rule and the domain of the specified composite functions.

(a) $f \circ g$ if $f(x) = x + 3$ and $g(x) = \frac{1}{x}$. $(f\circ g)(x) = \frac{1}{x} + 3$ domain $(f \circ g) = \{x \in \mathbb{R} \mid x \neq 0\}$

(b) $g \circ f$ if $f(x) = \frac{x}{x^2+1}$ and $g(x) = \sqrt{x}$. $\rightsquigarrow$ see previous pages for solution

(c) $f \circ g$ and $g \circ f$ if $f(x) = x + \frac{1}{x}$ and $g(x) = \frac{x+1}{x+2}$.

$$(f\circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x+2}\right) = \frac{x+1}{x+2} + \frac{x+2}{x+1} = \frac{(x+1)^2 + (x+2)^2}{(x+2)(x+1)} = \frac{x^2+6x+5}{x^2+3x+2}$$

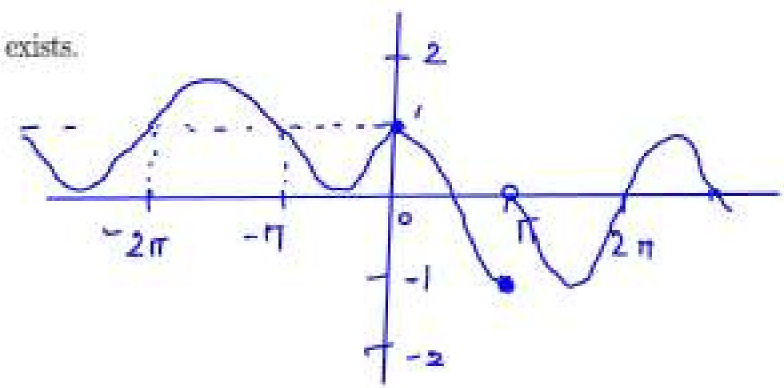
domain $(f\circ g) = \mathbb{R} \setminus \{-1,-2\}$

$$(g\circ f)(x) = g(f(x)) = g\left(x + \frac{1}{x}\right) = \frac{\left(x+\frac{1}{x}\right)+1}{\left(x+\frac{1}{x}\right)+2} = \frac{x^2+x+1}{x^2+2x+1}$$

domain $(g\circ f) = \mathbb{R} \setminus \{0,-1\}$

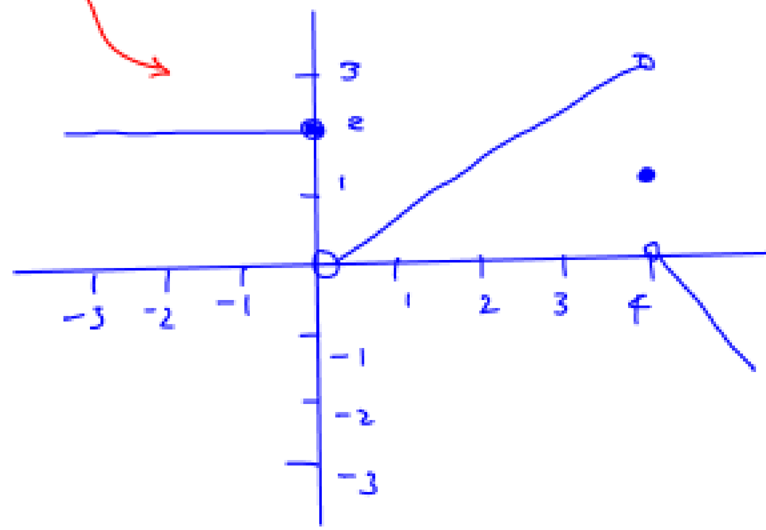
(13) Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

$$f(x) = \begin{cases} 1 + \sin x & \text{if } x < 0 \\ \cos x & \text{if } 0 \leq x \leq \pi \\ \sin x & \text{if } x > \pi \end{cases}$$



(14) Sketch the graph of an example of a function f that satisfies all of the given conditions.

(a) $\lim_{x\to 0^-} f(x) = 2$, $\lim_{x\to 0^+} f(x) = 0$, $\lim_{x\to 4^-} f(x) = 3$, $\lim_{x\to 4^+} f(x) = 0$ $f(0) = 2$, $f(4) = 1$.



(15) Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leq 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x\to 1^-} g(x)$ $= 1$

(ii) $\lim_{x\to 1} g(x)$ $= 1$

(iii) $g(1)$ $= 3$

(iv) $\lim_{x\to 2^-} g(x)$ $= -2$

(v) $\lim_{x\to 2^+} g(x)$ $= -1$

(vi) $\lim_{x\to 2} g(x)$ Does not exist

(16) Evaluate the limit, if it exists

(a) $\lim_{x\to -3} \dfrac{\sqrt{x^2+16}-5}{x+3}$ $\rightsquigarrow$ See previous page for solution

(b) $\lim_{x\to 3} \dfrac{x^2-x-6}{x-3}$ $\xrightarrow{\text{factorize}}$ $\dfrac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}}$ $\xrightarrow[x=3]{\text{substitute}}$ 5

(c) $\lim_{h\to 0} \dfrac{(x+h)^2-x^2}{h}$ $\xrightarrow[(x+h)^2]{\text{expand}}$ $\dfrac{\cancel{x^2}+2hx+h^2-\cancel{x^2}}{h}$ $\xrightarrow[\text{factor } h]{\text{Common}}$ $\dfrac{\cancel{h}(2x+h)}{\cancel{h}}$ $\xrightarrow[h\to 0]{}$ $2x$