MTH305 – Quiz 3 Name: SOLUTIONS

Question 1. Where are each of the following functions discontinuous? Justify your answer. At each of these numbers (of discontinuity), determine if f continuous from the right, from the left, or neither.
(2 points each)

(a) $f(x) = \dfrac{x^2 - x - 2}{x + 1}$

Since f is not defined at $x = -1$, f is discontinuous at $\boxed{x = -1}$

at $x = -1$, f is neither continuous from the right nor continuous from the left (since $f(-1)$ is not defined)

Note: f is a rational function with domain $\mathbb{R} \setminus \{-1\}$. Recall that rational functions are continuous on their domain.

(b) $f(x) = \begin{cases} \dfrac{x^2 - 9}{x + 3} & \text{if } x \neq -3 \\ 6 & \text{if } x = -3 \end{cases}$

Note that for $x \neq -3$, $f(x) = \dfrac{x^2 - 9}{x+3} = \dfrac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}} = x - 3.$

Hence, we have $\lim\limits_{x \to -3^-} f(x) = \lim\limits_{x \to -3^-} (x-3) = \lim\limits_{x \to -3^+} (x-3) = \lim\limits_{x \to -3^+} f(x) = -6.$

However, $f(-3) = 6$.

Since $\lim\limits_{x \to -3} f(x) \neq f(-3)$, f is not continuous at $\boxed{x = -3.}$

At $x = -3$, f is neither continuous from the left nor continuous from the right (Since $\lim\limits_{x \to -3^-} f(x) \neq f(-3)$ and $\lim\limits_{x \to -3^+} f(x) \neq f(-3)$).

(c) $f(x) = \begin{cases} -x-1 & \text{if } x < -1 \\ -x^2+1 & \text{if } -1 \le x \le 2 \\ x-2 & \text{if } x > 2 \end{cases}$

Since f is piecewise-polynomial, we investigate $x=-1$ and $x=2$ for points of discontinuity.

at $x=-1$ $\quad \lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-x-1) = -(-1)-1 = 1-1 = 0.$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (-x^2+1) = 0.$ Moreover, $f(-1) = -(-1)^2+1 = 0.$

Since $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) = f(-1)$, f is continuous at $x=-1$.

at $x=2$ $\quad \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (-x^2+1) = -4+1 = -3.$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-2) = 2-2 = 0.$

(extra work space)

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $\lim_{x \to 2} f(x)$ does not exist. Hence f is discontinuous at $\boxed{x=2.}$

Furthermore, at $x=2$, $f(2) = -2^2+1 = -3$.

Since $\lim_{x \to 2^-} f(x) = f(2)$, f is continuous from the left at $x=2$.

Question 2. Find the values of A and B so that the following function is continuous for all values of x. *(4 points)*.

$$f(x) = \begin{cases} Ax + \dfrac{B}{2} & \text{if } x \leq 1 \\ x^2 - 4Ax + \dfrac{5}{2}B & \text{if } 1 < x \leq 3 \\ 2 & \text{if } x > 3 \end{cases}$$

For f to be continuous for all x, we need f to be continuous at $x=1$ and $x=3$.

Therefore, at $x=1$, we require

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x).$$

$$\Rightarrow \quad A + \frac{B}{2} = 1 - 4A + \frac{5}{2}B$$

$$\Rightarrow \quad 5A - 2B = 1 \quad \text{——— (1)}$$

Similarly, at $x=3$, we require

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\Rightarrow \quad 9 - 12A + \frac{5}{2}B = 2$$

$$\Rightarrow \quad 12A - \frac{5}{2}B = 7 \quad \text{——— (2)}$$

We now solve the equations

$$5A - 2B = 1 \quad \text{——— (1)}$$

$$12A - \frac{5}{2}B = 7 \quad \text{——— (2)}$$

for A, B

$\frac{5}{2}$(1) gives $\quad \frac{25}{2}A - 5B = \frac{5}{2}$

2(2) gives $\quad 24A - 5B = 14$

Subtracting, we obtain $\left(\frac{25}{2} - 24\right)A = \left(\frac{5}{2} - 14\right)$

$$\Rightarrow \quad -\frac{23}{2}A = -\frac{23}{2} \quad \text{or} \quad \boxed{A = 1.}$$

Substituting in (1), we obtain $\quad 2B = 5(1) - 1 = 4 \quad$ or $\quad \boxed{B = 2.}$