MTH305 – Quiz 2 Name: SOLUTIONS

Question 1. Decide whether the geometric series below converge or diverge. Justify your answer. If the series converges, compute its sum. (*3 points*)

(a) $\pi - \pi + \pi - \pi + \pi - \pi + \ldots$

The common ratio is $r = -1$.

Since $|r| = 1$, this series <u>diverges</u>.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{3}\right)^n$

The series can be written as

$$\frac{\sqrt{2}}{3} + \left(\frac{\sqrt{2}}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^3 + \ldots\ldots$$

the first term, $a = \frac{\sqrt{2}}{3}$

the common ratio, $r = \frac{\sqrt{2}}{3}$

Since $|r| < 1$, this series <u>converges</u>

Sum, $\displaystyle\sum_{n=1}^{\infty} \left(\frac{\sqrt{2}}{3}\right)^n = \frac{a}{1-r} = \frac{\sqrt{2}/3}{1 - \sqrt{2}/3} = \underline{\frac{\sqrt{2}}{3-\sqrt{2}}}$

Question 2. Compute the sum $S = -16 + 8 - 4 + \cdots - \frac{1}{64}$ without using a calculator. (*2 points*)

The terms of this sum form a geometric sequence with

first term, $a = -16$ and

Common ratio, $r = -\frac{1}{2}$

Using $x_n = a r^{n-1}$, we get

$$-\frac{1}{64} = -16\left(-\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow \frac{(-1)}{2^6} = \frac{16(-1)(-1)^{n-1}}{2^{n-1}}$$

$$\Rightarrow \frac{(-1)}{2^6} = \frac{2^4(-1)^n}{2^{n-1}}$$

$$\Rightarrow \frac{(-1)}{2^6} = \frac{(-1)^n}{2^{n-5}}$$

$$\Rightarrow \underline{n = 11}$$

Therefore, $S = \frac{a(1-r^n)}{1-r}$

$$= \frac{(-16)\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)}$$

$$= \frac{-2^4\left(1+\frac{1}{2^{11}}\right)}{1+\frac{1}{2}}$$

$$S = \frac{-2^4\left(\frac{2^{11}+1}{2^{11}}\right)}{3/2} = \underline{\frac{-(2^{11}+1)}{3\cdot 2^6}}$$

Question 3. Find the assignment rule (or law/formula) and the domain of the specified composite functions below: (*3 points*)

$$f \circ g \quad \text{and} \quad g \circ f, \quad \text{if} \quad f(x) = \frac{x}{x^2 - 6} \quad \text{and} \quad g(x) = \sqrt{x}.$$

Note: $\text{domain}(f) = \mathbb{R} \setminus \{\sqrt{6}, -\sqrt{6}\}$

$\text{domain}(g) = [0, \infty)$

$(f \circ g)(x) = f(g(x)) = f(\sqrt{x})$

$= \dfrac{\sqrt{x}}{x-6}$

$\text{domain}(f \circ g) = [0, \infty) \setminus \{6\}$

Note:

- want $x \in \text{domain}(g)$
  $\Rightarrow x \in [0, \infty)$
- want $g(x) \in \text{domain}(f)$
  $\Rightarrow \sqrt{x} \neq \sqrt{6}$ and $\sqrt{x} \neq -\sqrt{6}$
  $\Rightarrow x \neq 6$

$(g \circ f)(x) = g(f(x)) = g\left(\dfrac{x}{x^2-6}\right)$

$= \sqrt{\dfrac{x}{x^2-6}}$

$\text{domain}(g \circ f) = (-\sqrt{6}, \boxed{0}] \cup (\sqrt{6}, \infty)$

Note:

- want $x \in \text{domain}(f)$
  $\Rightarrow x \in \mathbb{R} \setminus \{\sqrt{6}, -\sqrt{6}\}$
- want $f(x) \in \text{domain}(g)$
  $\Rightarrow \dfrac{x}{x^2-6} \geq 0$

this holds when $x > \sqrt{6}$ and $-\sqrt{6} < x \leq 0$.

Question 4. For each of the following, determine if the given equation describes y as a **function** of x. For the one(s) that you identify to be (a) function(s), determine the domain. (*2 points*)

(a) $y = \dfrac{1}{\sqrt{x^2 - 3x}}$

Yes, this describes a function.

For each x-value (input), there is a unique (output) y-value.

Domain $= \mathbb{R} \setminus [0, 3]$

Note:

We have $y = \dfrac{1}{\sqrt{x^2-3x}}$

$= \dfrac{1}{\sqrt{x(x-3)}}$

Clearly this is not defined for $x=0$ and $x=3$.

Furthermore, for $0 < x < 3$, $x(x-3) < 0$, and, hence $\sqrt{x(x-3)}$ is not defined

(b) $x - y^2 = 4$

No, this does not describe a function.

For example, $y=1$ and $y=-1$ both satisfy the given equation when $x=5$.

Therefore, there are multiple (output) y-values for the same (input) x-value.