**Question 1.** Determine direct and recursive formulas for the given sequences below. Make sure to give the value of the sequence's first term. (*2 points each*)

(a) An arithmetic sequence $\{x_n\}_{n\geq 1}$ with $x_4 = 3$ and a common difference of $\frac{1}{3}$.

We know that for an arithmetic sequence

$$x_n = x_1 + (n-1)d.$$

Therefore, $x_4 = 3 = x_1 + 3\left(\frac{1}{3}\right)$, or $x_1 = 2$.

Direct formula: $x_n = 2 + \frac{1}{3}(n-1)$, $n \geq 1$.

Recursive formula: $x_1 = 2$, $x_{n+1} = x_n + \frac{1}{3}$, $n \geq 1$.

(b) A geometric sequence $\{x_n\}_{n\geq 1}$ with $x_3 = 1$ and a common ratio of $\frac{1}{2}$.

For a geometric sequence, $x_n = x_1 r^{n-1}$.

For $n=3$, we get $x_3 = 1 = x_1 \left(\frac{1}{2}\right)^2$, or $x_1 = 4$.

Direct formula: $x_n = 4\left(\frac{1}{2}\right)^{n-1}$, $n \geq 1$.

Recursive formula: $x_1 = 4$, $x_{n+1} = \frac{1}{2} x_n$, $n \geq 1$.

**Question 2.** Determine if the following statements are true or false. No justification required. (*2 points*)

(a) The sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ is a bounded sequence. TRUE

(note: bound $M = 1$)

(b) The sequence $\{(-1)^n\}_{n\in\mathbb{N}}$ is a convergent sequence. FALSE

**Question 3.** Determine if the given sequence is convergent or divergent; if convergent, determine its limit. Briefly justify your answers. (*2 points each*)

(a) $x_n = 1 + \left(\frac{1}{3}\right)^n$

CONVERGENT

using sequence properties,

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\left(\frac{1}{3}\right)^n\right) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{1}{3^n} = 1 + \lim_{n\to\infty}\frac{1}{3^n}.$$

Since $-\frac{1}{n} \leq \frac{1}{3^n} \leq \frac{1}{n}$, applying the sandwich theorem tells us that $\lim_{n\to\infty}\frac{1}{3^n} = \lim_{n\to\infty}\frac{1}{n} = 0$.

Therefore $L = \lim_{n\to\infty} x_n = 1$.

(b) $a_n = (-1)^n\, 2n$

DIVERGENT

We have $a_n = \begin{cases} 2n, & \text{if } n \text{ is even,} \\ -2n, & \text{if } n \text{ is odd.}\end{cases}$

there is no $L$ such that $L = \lim_{n\to\infty} a_n$. This can be seen as follows:

Case ① $(L=0)$ Consider the interval $(-1,1)$. We see that $a_n \notin (-1,1)$ for all $n \geq 1$.

Case ② $(L>0)$ — " — $(0, 2L)$. — " — $a_n \notin (0, 2L)$ for all odd $n$.

Case ③ $(L<0)$ — " — $(2L, 0)$ — " — $a_n \notin (2L, 0)$ for all even $n$.

In all cases, there are an infinite number of terms of the sequence which are not contained in the respective intervals.

Hence, there is no $L$ such that $L = \lim_{n\to\infty} a_n$; therefore the sequence diverges.