# INTEGRATION (CHAPTER 5)

**Defn** A function $F$ is called an <u>antiderivative</u> of a function $f$ if $F'(x) = f(x)$ for all $x$ in the domain of $f$.

The process of finding a function from its derivative is called <u>antidifferentiation</u>.

**Thm** Let $f$ and $F$ be two functions defined on the interval $I$ with $F$ being an antiderivative of $f$. If $C$ is a constant, then the function $F(x) + c$ defined for all $x$ in the interval $I$ is also an antiderivative of $f$. Conversely any antiderivative of $f$ is of this form

**Notation** Antidifferentiation is also called <u>integration</u>.

A function is said to be <u>integrable</u> if it has antiderivatives.

The family of all antiderivatives of a given function $f$ is called the <u>Indefinite integral</u> of $f$; denoted $\int f(x)\,dx$

which is read as "integral of $f$ w.r.t. $x$" or "integral of $f(x)\,dx$".

We have $\int f(x)\,dx = \{ F + c : c \in \mathbb{R} \}$

For simplicity we write

antiderivative

integral sign $\quad \int f(x)\,dx = F(x) + C \quad$ constant of integration

differential

Integrand $\qquad$ variable of integration

## Rules of Integration

1. $\int k f(x)\,dx = k \int f(x)\,dx$ (constant multiple rule)

2. $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ (Sum|Difference rule)

3. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c \quad (n \neq -1)$ ( Simple power rule)

4. $\int e^x\,dx = e^x + C$ ( -"- exponential rule)

5. $\int e^{kx}\,dx = \dfrac{1}{k} e^{kx} + C \quad (k \neq 0)$

Exercises

① Find the integrand domain, then compute the indefinite integral. Use differentiation to justify your answers.

(a) $\int x^7 dx$

(b) $\int 5 dx$

(c) $\int (1+\sqrt{2}) dx$

(d) $\int t^{5/2} dt$

(e) $\int (3+x^{-3}) dx$

(f) $\int \left(t^2 - \frac{1}{t\sqrt{t}}\right)(t+1) dt$

(g) $\int 3e^t dt$

(d) $\int t^{5/2} dt$

* the domain is all real numbers ($\mathbb{R}$)

* Using the power rule,

$$\int t^{5/2} dt = \frac{t^{(5/2+1)}}{(5/2+1)} + C$$

$$= \frac{2}{7} t^{7/2} + C.$$

* verification $\frac{d}{dt}\left(\frac{2}{7} t^{7/2} + C\right) = t^{5/2}$

① Find the integrand domain, then compute the indefinite integral.

Use differentiation to justify your answers.

(a) $\int x^7\,dx$

(b) $\int 5\,dx$

(c) $\int (1+\sqrt{2})\,dx$

(d) $\int t^{5/2}\,dt$

(e) $\int (3+x^{-3})\,dx$

(f) $\int \left(t^2-\frac{1}{t\sqrt{t}}\right)(t+1)\,dt$

(g) $\int 3e^t\,dt$

(e) $\int (3+x^{-3})\,dx$

* domain of integrand is all real numbers excluding $x=0$; or, $(-\infty,0)\cup(0,\infty)$

* for $x\in(-\infty,0)$

$$\int (3+x^{-3})\,dx = \int 3\,dx + \int x^{-3}\,dx \quad \text{(sum rule)}$$

$$= 3\int x^0\,dx + \int x^{-3}\,dx \quad \text{(const. mult. rule)}$$

$$= 3\,\frac{x^{(0+1)}}{(0+1)} + \frac{x^{(-3+1)}}{(-3+1)} + c \quad \text{(power rule)}$$

$$= 3x - \frac{1}{2}x^{-2} + C$$

these are different constants

Similarly, for $x\in(0,\infty)$, $\int (3+x^{-3})\,dx = 3x-\frac{1}{2}x^{-2}+D$

Why? compute derivative of

$$F(x)=\begin{cases} 3x-\frac{1}{2}x^{-2}+C & \text{if } x<0 \\ 3x-\frac{1}{2}x^{-2}+D & \text{if } x>0\end{cases}$$

and observe that $F'(x) = 3+x^{-3}$.

① Find the integrand domain, then compute the indefinite integral. Use differentiation to justify your answers.

(a) $\int x^7 \, dx$

(b) $\int 5 \, dx$

(c) $\int (1+\sqrt{2}) \, dx$

(d) $\int t^{5/2} \, dt$

(e) $\int (3+x^{-3}) \, dx$

(f) $\int \left(t^2 - \frac{1}{t\sqrt{t}}\right)(t+1) \, dt$

(g) $\int 3e^t \, dt$

(f) $\int \left(t^2 - \frac{1}{t\sqrt{t}}\right)(t+1) \, dt$

* domain of integrand is $(0,\infty)$

* $$\int \left(t^2 - \frac{1}{t\sqrt{t}}\right)(t+1) \, dt = \int \left(t^3 - \frac{1}{\sqrt{t}} + t^2 - \frac{1}{t\sqrt{t}}\right) dt$$

$$= \int t^3 \, dt - \int t^{-1/2} \, dt + \int t^2 \, dt - \int t^{-3/2} \, dt$$

(Sum/difference rule)

$$= \frac{t^4}{4} + c_1 - \left(\frac{t^{1/2}}{1/2} + c_2\right) + \frac{t^3}{3} + c_3 - \left(\frac{t^{-1/2}}{-1/2} + c_4\right)$$

(power rule)

$$= \frac{t^4}{4} - 2t^{1/2} + \frac{t^3}{3} + 2t^{-1/2} + c$$

where $c := c_1 - c_2 + c_3 - c_4$.

* verify by computing $\frac{d}{dt}\left(\frac{t^4}{4} - 2t^{1/2} + \frac{t^3}{3} + 2t^{-1/2} + c\right)$

② Find the antiderivative $F(x)$ of the function $f(x) = e^x + x - 1$ that satisfies $F(0) = 6$.

$$F(x) = \int f(x)\,dx$$

$$= \int (e^x + x - 1)\,dx$$

$$= \int e^x dx + \int x\,dx - \int 1.\,dx \quad \text{(by sum/difference rule/property)}$$

$$= (e^x + c_1) + \left(\frac{x^2}{2} + c_2\right) - (x + c_3)$$

integration constants

$$= e^x + \frac{x^2}{2} - x + c \qquad \text{where } c := c_1 + c_2 - c_3$$

Since we are given $F(0) = 6$,

we have $F(0) = e^0 + \frac{0^2}{2} - 0 + c = 6$

$$\Rightarrow \quad 1 + c = 6$$

$$\Rightarrow \quad c = 5$$

Therefore, the desired antiderivative is $F(x) = e^x + \frac{x^2}{2} - x + 5$.