SOLUTIONS

Homework 6; Due Tuesday, 04/25/2017

Quick Answer Questions. No work needed. No partial credit available.

Question 1. (*Fill in the blanks and multiple choice questions*)

(I) Let $G(x) = \int_x^3 e^{t^2}\,dt$, then fill in the blanks:
$G(x)$ is the antiderivative of $g(t) =$ $-e^{t^2}$ that takes the value 0 at $x =$ 3.

(II) The antiderivative $F(x)$ of $f(x) = x^2 - e^x - 1$ that satisfies $F(-1) = e$ is (*choose the correct answer(s)*)

(a) $F(x) = x^3/3 - e^x - x$

(b) $F(x) = 2x - xe^{x-1} + (e + 2 - e^{-2})$

(c) $F(x) = 2x - e^x + C$

(d) $\boxed{F(x) = x^3/3 - e^x - x + (e + 1/e - 2/3)}$

(e) none of the above

(III) The antiderivative $F(x)$ of $f(x) = -2/x^3$ that satisfies $F(-2) = 0$ is (*choose the correct answer(s)*)

(a) $F(x) = 1/x^2 - 1/4$

(b) $F(x) = 1/x^2 + C$

(c) $\int_{-2}^{x} -2/t^3 dt$

(d) $F(x) = 1/x^2$

(e) $\boxed{F(x) = \begin{cases} 1/x^2 - 1/4 & \text{if } x < 0 \\ 1/x^2 - C & \text{if } x > 0 \end{cases} \text{ where } C \text{ is any real number.}}$

note: $\mathrm{dom}(f) = \underbrace{(-\infty, 0)}_{\text{interval 1}} \cup \underbrace{(0, \infty)}_{\text{interval 2}}$
there is a constant of integration corresponding to each interval

(IV) (*Fill in the blanks*) Given $\int_1^2 f(x)dx = 2$, $\int_2^3 f(x)dx = 4$, $\int_1^2 g(x)dx = 1$, and $\int_2^3 g(x)dx = -1$,

(a) $\int_1^3 f(x)dx =$ 6. $\int_1^3 f(x)dx = \int_1^2 f(x)dx + \int_2^3 f(x)dx$

(b) $\int_1^3 g(x)dx =$ 0. $\int_1^3 g(x)dx = \int_1^2 g(x)dx + \int_2^3 g(x)dx$

(c) $\underbrace{\int_1^3 [5f(x) - 2g(x)]dx}_{I} =$ 30. $I = 5\int_1^3 f(x)dx - 2\int_1^3 g(x)dx = 5(6) - 2(0) = 30$

Longer Questions. Provide complete justifications for your responses.

Question 2. In the following exercises, find the integrand's domain, then compute the indefinite integral. Use differentiation to justify your answers.

(a) $\displaystyle\int (x + e^x)dx$.

(b) $\displaystyle\int (-t^3 + 1)dt$.

(see following pages)

(c) $\displaystyle\int \left(2t + \frac{1}{t\sqrt{t}}\right)(t^2 + t)dt$.

Question 3. A diver jumps from a cliff with an upward initial velocity. The cliff is 60 feet above the sea.

(a) What is the diver's initial velocity if he reaches the maximal height after 0.25 second?

(b) What is the maximal height reached by the diver?

(c) After how many seconds does the diver hit the water?

(see following pages)

(d) What is the diver's velocity at impact?

Question 4. In the following exercises, compute the definite integral $\displaystyle\int_a^b f(x)dx$. At each step, write down the properties of definite integrals used.

(a) $f(x) = (x^2 - 1)(x^4 + x^3), \quad a = 1, b = 2$

(b) $f(x) = 2|x| + 1, \quad a = -1, b = 4$

(see following pages)

(c) $f(x) = \sqrt{x}(x^2 - 1), \quad a = 0, b = 1$

Question 5. In the following exercises, compute the given integral. Use differentiation to justify your answers.

(a) $\displaystyle\int x(x^2 + 5)\sqrt{x^2 + 5}\, dx$.

(b) $\displaystyle\int (x + 1)e^{x^2+2x-2}\, dx$.

(c) $\displaystyle\int_0^1 \frac{\sqrt{x}}{(2 + \sqrt{x^3})^2}\, dx$.

#2 (a) $\int (x+e^x)\,dx$

* the integrand is $f(x) = x + e^x$. Domain of integrand is $\mathbb{R}$

* $\int (x+e^x)\,dx = \int x\,dx + \int e^x\,dx$ (Sum property)

$= \left(\frac{x^2}{2} + c_1\right) + \left(e^x + c_2\right)$ (power rule, exponential rule)

$\int (x+e^x)\,dx = \frac{x^2}{2} + e^x + C$ where $C = c_1 + c_2$; here $C \in \mathbb{R}$

* Verification: $\frac{d}{dx}\left(\frac{x^2}{2} + e^x + C\right) = \frac{d}{dx}\left(\frac{x^2}{2}\right) + \frac{d}{dx}(e^x) + \frac{d}{dx}(C)$

$= \frac{1}{2}(2x) + e^x + 0$

$= x + e^x$ (integrand)

(b) $\int (-t^3+1)\,dt$

* domain of integrand is all real numbers

* $\int (-t^3+1)\,dt = -\int t^3\,dt + \int 1\,dt$ (Sum; const. multiple rule)

$= -\frac{t^4}{4} + c_1 + t + c_2$ (power rule)

$= -\frac{t^4}{4} + t + C$ where $C = c_1 + c_2$ $(C \in \mathbb{R})$

* Verification: $\frac{d}{dt}\left(-\frac{t^4}{4} + t + C\right) = \frac{d}{dt}\left(-\frac{t^4}{4}\right) + \frac{d}{dt}(t) + \frac{d}{dt}(C)$

$= -\frac{1}{4}(4t^3) + 1 + 0$

$= -t^3 + 1$ (integrand)

#2 (c) $\int \left(2t + \frac{1}{t\sqrt{t}}\right)(t^2 + t)\,dt$

Rewriting, we have

$$I = \int \left(2t + \frac{1}{t\sqrt{t}}\right)(t^2+t)\,dt = \int \left(2t^3 + \frac{t}{\sqrt{t}} + 2t^2 + \frac{1}{\sqrt{t}}\right)dt$$

$$= \int \Big(\underbrace{2t^3 + t^{1/2} + 2t^2 + t^{-1/2}}_{\text{integrand}}\Big)dt$$

* domain of integrand is $(0, \infty)$

* $I = \int \left(2t^3 + t^{1/2} + 2t^2 + t^{-1/2}\right)dt = 2\int t^3 dt + \int t^{1/2} dt + 2\int t^2 dt + \int t^{-1/2} dt$

(sum, const. mult. rules)

$$= 2\left(\frac{t^4}{4} + c_1\right) + \left(\frac{t^{3/2}}{3/2} + c_2\right) + 2\left(\frac{t^3}{3} + c_3\right) + \left(\frac{t^{1/2}}{1/2} + c_4\right)$$

(power rule)

$$= \frac{t^4}{2} + \frac{2}{3}t^{3/2} + \frac{2}{3}t^3 + 2t^{1/2} + C$$

* verification: $\frac{d}{dt}\left(\frac{t^4}{2} + \frac{2}{3}t^{3/2} + \frac{2}{3}t^3 + 2t^{1/2} + C\right)$

$$= \frac{1}{2}\frac{d}{dt}(t^4) + \frac{2}{3}\frac{d}{dt}(t^{3/2}) + \frac{2}{3}\frac{d}{dt}(t^3) + 2\frac{d}{dt}(t^{1/2}) + \frac{d}{dt}(C)$$

$$= \frac{1}{2}(4t^3) + \frac{2}{3}\left(\frac{3}{2}t^{1/2}\right) + \frac{2}{3}(3t^2) + 2\left(\frac{1}{2}t^{-1/2}\right) + 0$$

$$= 2t^3 + t^{1/2} + 2t^2 + t^{-1/2} = \text{integrand}$$

#3

(a) * Let initial velocity $= v_0$ (in ft/s)

* at maximal height, (final) velocity $= v_f = 0$ ft/s
* time taken, $\Delta t = \frac{1}{4}$ s
* Diver subject to (const.) $\underbrace{\text{acceleration}}_{\text{rate of change of velocity}}$ due to gravity; i.e., $a = -32$ ft/s^2

Hence $a = \dfrac{\Delta v}{\Delta t} = \dfrac{v_f - v_0}{\Delta t}$

$$\Rightarrow -32 = \frac{0 - v_0}{\frac{1}{4}}$$

$$\Rightarrow v_0 = 32\left(\frac{1}{4}\right) = \boxed{8 \text{ ft/s}}$$

(b) For upward portion of dive,

velocity $v(t) = v_0 + at$ or, $v(t) = 8 - 32t$ (in ft/s)

note: velocity constantly changes due to force of gravity

Distance covered in upward portion of dive (in $\frac{1}{4}$ s)

$$s_1 = \int_0^{\frac{1}{4}} v(t)\,dt = \int_0^{\frac{1}{4}} (8 - 32t)\,dt$$

$$= 8\int_0^{\frac{1}{4}} dt - 32\int_0^{\frac{1}{4}} t\,dt$$

$$= 8\left(1\Big|_0^{\frac{1}{4}}\right) - 32\left(\frac{t^2}{2}\Big|_0^{\frac{1}{4}}\right)$$

$$= 8\left(\frac{1}{4}\right) - 16\left(\frac{1}{4^2}\right) = 2 - 1 = \boxed{1 \text{ ft}}$$

#3 (c)

since diver is falling

* the diver is now $S = \underbrace{-(60}_{\text{cliff height}} + \underbrace{1)}_{\text{jump height}} = -61$ ft above water —— (*)

* at maximal height, (initial) velocity $v_i = 0$ ft/s

* as the diver falls velocity changes as per

$$v(t) = v_i + \underbrace{a}_{\text{acceleration due to gravity}}t \qquad (\text{in ft/s})$$

$$v(t) = 0 - 32t \quad \text{——} \quad (**)$$

* distance covered in T secs (counting from maximal height)

$$S = \int_0^T v(t)\,dt = \int_0^T -32t\,dt$$

$$= -32\int_0^T t\,dt = -32\left(\frac{t^2}{2}\Big|_0^T\right) = -16T^2 \text{ ft}$$

Since we know $S = -61$ ft (see (*)), we have

$$-61 = -16T^2 \quad \text{or} \quad T = \sqrt{\frac{61}{16}} \boxed{\approx 1.953 \text{ Secs}}$$

Diver hits water after $\underbrace{0.25}_{\text{upward jump}} + \underbrace{1.953}_{\text{fall}}$ $\boxed{\approx 2.203 \text{ sec}}$

* velocity at impact

Using (**), $v(T) = v(1.953) = -32(1.953) \approx \boxed{-62.482 \text{ ft/s.}}$

#4 (a) $\int_1^2 (x^2-1)(x^4+x^3)\,dx$

$$= \int_1^2 (x^6 - x^4 + x^5 - x^3)\,dx \quad \text{(algebral sinplification)}$$

$$= \int_1^2 x^6 dx - \int_1^2 x^4 dx + \int_1^2 x^5 dx - \int_1^2 x^3 dx \quad \text{(sum|differenc rule)}$$

$$= \frac{x^7}{7}\Big|_1^2 - \frac{x^5}{5}\Big|_1^2 + \frac{x^6}{6}\Big|_1^2 - \frac{x^4}{4}\Big|_1^2 \quad \text{(power rule)}$$

$$= \frac{1}{7}(2^7-1) - \frac{1}{5}(2^5-1) + \frac{1}{6}(2^6-1) - \frac{1}{4}(2^4-1)$$

$$= \frac{127}{7} - \frac{31}{5} + \frac{63}{6} - \frac{15}{4}$$

$$= \frac{7620 - 2604 + 4410 - 1575}{420} = \boxed{\frac{7851}{420} \approx 18.693}$$

(b) $\int_{-1}^4 (2|x|+1)\,dx = 2\int_{-1}^4 |x|\,dx + \int_{-1}^4 1\,dx$ (sum rul)

$$= 2\left[\int_{-1}^0 (-x)\,dx + \int_0^4 x\,dx\right] + \int_{-1}^4 1.\,dx \quad \text{(prop. \#23(c), def}^n\text{ of } |x|)$$

$$= -2\int_{-1}^0 x\,dx + 2\int_0^4 x\,dx + \int_{-1}^4 dx$$

$$= -2\left(\frac{x^2}{2}\Big|_{-1}^0\right) + 2\left(\frac{x^2}{2}\Big|_0^4\right) + \left(x\Big|_{-1}^4\right)$$

$$= -1(0-1) + 1(16-0) + (4-(-1)) -$$

$$= 1 + 16 + 5 = \boxed{22}$$

#4 (c) $\int_0^1 \sqrt{x}(x^2-1)\,dx = \int_0^1 (x^{5/2} - x^{1/2})\,dx$

$= \int_0^1 x^{5/2}\,dx - \int_0^1 x^{1/2}\,dx$ (difference rule)

$= \left(\frac{x^{7/2}}{7/2}\Big|_0^1\right) - \left(\frac{x^{3/2}}{3/2}\Big|_0^1\right)$ (power rule)

$= \frac{2}{7}(1-0) - \frac{2}{3}(1-0)$

$= \frac{2}{7} - \frac{2}{3} = \boxed{\frac{-8}{21}}$

#5 (a) $I = \int x(x^2+5)\sqrt{x^2+5}\,dx$

Let $u(x) = x^2+5$. Then $u'(x) = \frac{du}{dx} = 2x$, or $\frac{1}{2}\,du = x\,dx$

Therefore $I = \int u\sqrt{u}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{3/2}\,du$

$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2}\right) + C$

$\boxed{I = \frac{1}{5}(x^2+5)^{5/2} + C}$

* verification: $\frac{d}{dx}\left[\frac{1}{5}(x^2+5)^{5/2} + C\right] = \frac{1}{5}\left(\frac{5}{2}\right)(x^2+5)^{3/2} \cdot \frac{d}{dx}(x^2+5) + 0$

(chain rule)

$= \frac{1}{2}(x^2+5)^{3/2}(2x)$

$= x(x^2+5)^{3/2}$

$= x(x^2+5)\sqrt{x^2+5}$ = integrand

#5 (b) $I = \int (x+1)\, e^{x^2+2x-2}\, dx$

Let $x^2+2x-2 = u(x)$.

Then $\frac{du}{dx} = u'(x) = 2x+2 = 2(x+1)$.

$\Rightarrow \frac{1}{2}\, du = (x+1)\, dx$

Therefore $I = \int e^u \left(\frac{1}{2} du\right) = \frac{1}{2}\int e^u du = \frac{1}{2} e^u + C$

$$\boxed{I = \frac{1}{2} e^{(x^2+2x-2)} + C}$$

(c) $I = \int_0^1 \frac{\sqrt{x}}{(2+\sqrt{x^3})^2}\, dx$

Let $u(x) = 2+\sqrt{x^3} = 2+x^{3/2}$.

then $u'(x) = \frac{du}{dx} = 0 + \frac{3}{2}x^{1/2}$, or $\frac{2}{3} du = \sqrt{x}\, dx$.

In addition, when $x=0$, $u = 2+\sqrt{0^3} = 2$

$x=1$, $u = 2+\sqrt{1^3} = 3$

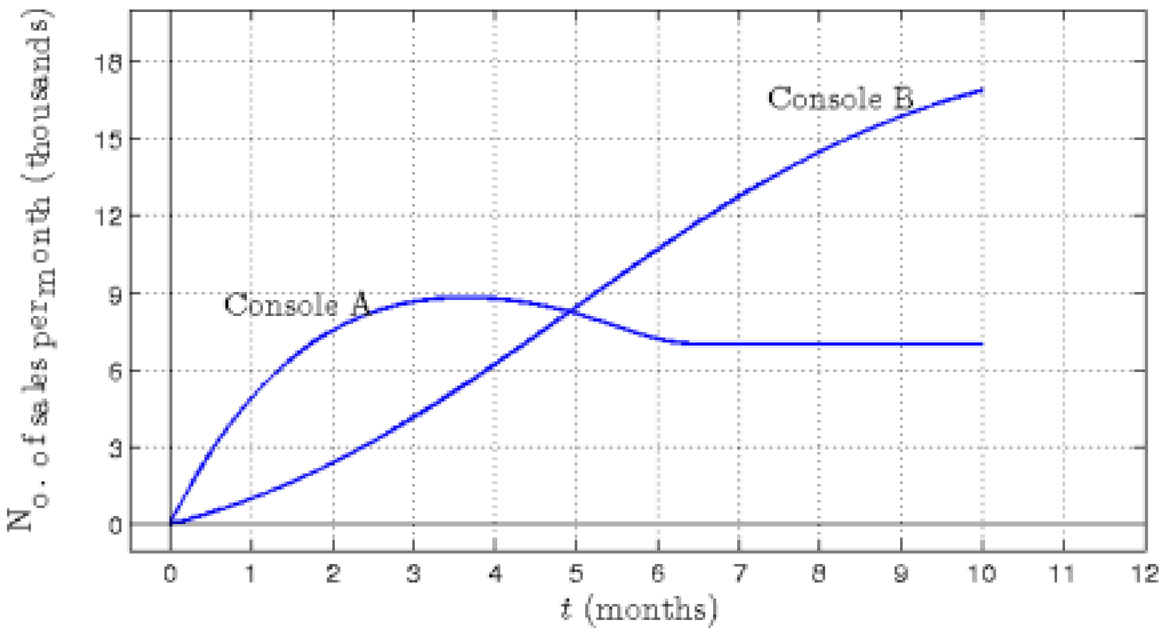
Substituting in I, we get

$$I = \int_2^3 \frac{\frac{2}{3}du}{u^2} = \frac{2}{3}\int_2^3 u^{-2} du = \frac{2}{3}\left(\frac{u^{-1}}{-1}\Big|_2^3\right)$$

$$= -\frac{2}{3}\left(\frac{1}{3}-\frac{1}{2}\right)$$

or $\boxed{I = \frac{1}{9}}$

Question 6. The figure below shows the number of sales per month (in thousands) of two competing game consoles.



(a) Which console has the most total sales after 5 months? After 10 months? How did you arrive at your answer?

(b) At approximately what times (if any) have they sold roughly equal total amounts?

(c) Approximately how many total units of console A have been sold in 10 months?

(a) After 5 months, console A has the most sales.
After 10 months, console B has the most sales.
The total sales is given by the area under each curve. In the interval $0 \le t \le 5$, the area under the graph of console A is greater than the area under the graph of console B. Similar arguments hold at 10 months.

(b) By counting grid squares, we find that at about 8 months, the areas under both curves are approximately equal.

(c) There are ≈ 23 grid squares between the graph of console A and the t-axis in $0 \le t \le 10$.
The area of each grid square is $3000 \left(\text{sales}/\text{month}\right) \cdot 1(\text{month}) = 3000$ units
Hence, console A sold about 69000 units in 10 months.