Homework 3 - Key

#1 Study the continuity of f on the real line (i.e, identify intervals of continuity, point(s) of discontinuity and the reason for said discontinuity)

(a) $$f(x) = \begin{cases} x^3 - 2 & \text{if } x \leq 2 \\ x - 2x^2 - 1 & \text{if } x > 2 \end{cases}$$

In each of the intervals $(-\infty, 2)$ and $(2, \infty)$, f is a polynomial, and, hence continuous.

We now investigate continuity of f at $x = 2$.

We have (*) $f(2)$ is defined with $f(2) = 2^3 - 2 = 6$.

(*) $$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 2) = 2^3 - 2 = 6.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 2x^2 - 1) = 2 - 2(2^2) - 1 = -7.$$

Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, $\lim_{x \to 2} f(x)$ does not exist

It follows then that f is discontinuous at $x = 2$

(b)
$$f(x) = \begin{cases} x^2 - \frac{1}{2}x + 1 & \text{if } x \leq 1 \\ \frac{3x}{x+1} & \text{if } x > 1 \end{cases}$$

In the interval $(-\infty, 1)$, $f(x) = x^2 - \frac{1}{2}x + 1$. this is a polynomial; hence f is continuous in $(-\infty, 1)$.

In the interval $(1, \infty)$, $f(x) = \frac{3x}{x+1}$. This is a rational function. Since $(1, \infty) \in \text{dom}\left(\frac{3x}{x+1}\right)$, it follows that f is continuous on $(1, \infty)$.

We now investigate continuity at $x = 1$. We see that

(*) $f(1)$ is defined, with $f(1) = 1^2 - \frac{1}{2}(1) + 1 = \frac{3}{2}$.

(*) $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 - \frac{1}{2}x + 1) = 1^2 - \frac{1}{2}(1) + 1 = \frac{3}{2}$.

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \left(\frac{3x}{x+1}\right) = \frac{3(1)}{(1)+1} = \frac{3}{2}$.

Hence, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} f(x) = \frac{3}{2}$.

(*) $\lim_{x \to 1} f(x) = f(1) = \frac{3}{2}$.

Hence f is continuous at $x = 1$. Therefore f is continuous for all $x \in \mathbb{R}$.

#2 Find the values of a for which the function f is continuous on $\mathbb{R}$.

(a) $$f(x) = \begin{cases} x-a & \text{if } x \leq 0 \\ x^2+1 & \text{if } x > 0 \end{cases}$$

Since f is piecewise-polynomial, f is continuous on $(-\infty, 0)$ and $(0, \infty)$.

For continuity at $x=0$, we require $\lim\limits_{x\to 0} f(x) = f(0)$.

Equivalently, we require $\lim\limits_{x\to 0^+} f(x) = f(0)$

(Since we require $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x)$)

$$\Rightarrow \lim_{x\to 0^+} (x^2+1) = (0) - a$$

$$\Rightarrow 0 + 1 = -a$$

$$\Rightarrow \boxed{a = -1.}$$

f is continuous for all $x \in \mathbb{R}$ when $a = -1$.

(b) $f(x) = \begin{cases} -2x - a^2 & \text{if } x \leq 1 \\ 1 - 4ax & \text{if } x > 1 \end{cases}$

f is piecewise-polynomial. Therefore, it is continuous on the intervals $(-\infty, 1)$, $(1, \infty)$.

For continuity at $x = 1$, we require $\lim_{x \to 1} f(x) = f(1)$, or, equivalently,

$$\lim_{x \to 1^+} f(x) = f(1) \qquad \left(\text{Since we require } \lim_{x \to 1} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x)\right)$$

$$\Rightarrow \quad 1 - 4a(1) = -2(1) - a^2$$

$$\Rightarrow \quad a^2 - 4a + 3 = 0$$

$$\Rightarrow \quad (a-3)(a-1) = 0$$

$$\Rightarrow \quad \boxed{a = 3 \text{ or } a = 1.}$$

f is continuous for all $x \in \mathbb{R}$ when $a = 3$ or $a = 1$.

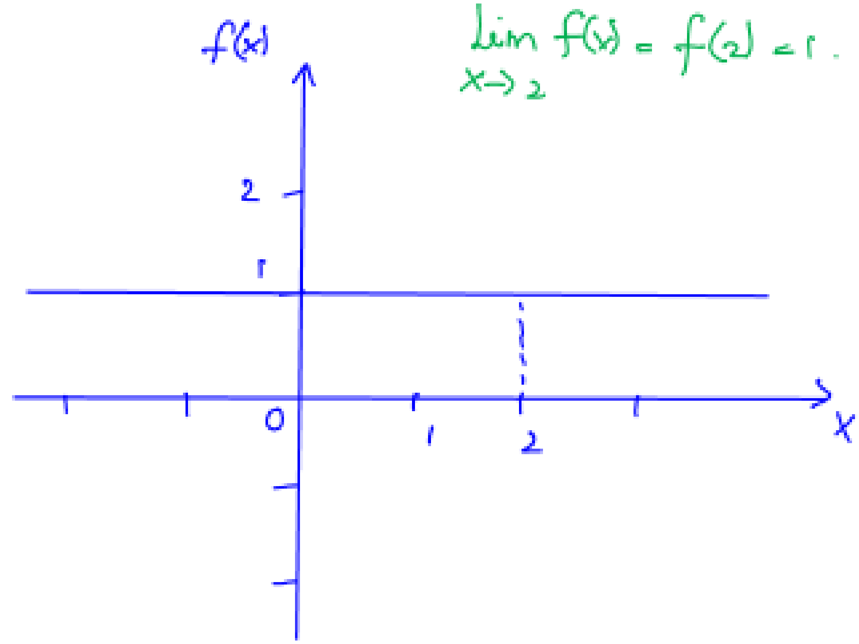
#3 Give an example of a function f as specified

(a) f is continuous at $x=2$.

$\lim_{x \to 2} f(x) = f(2) = 1$.

Here is one such f

$f(x) = x$ for all $x \in \mathbb{R}$.

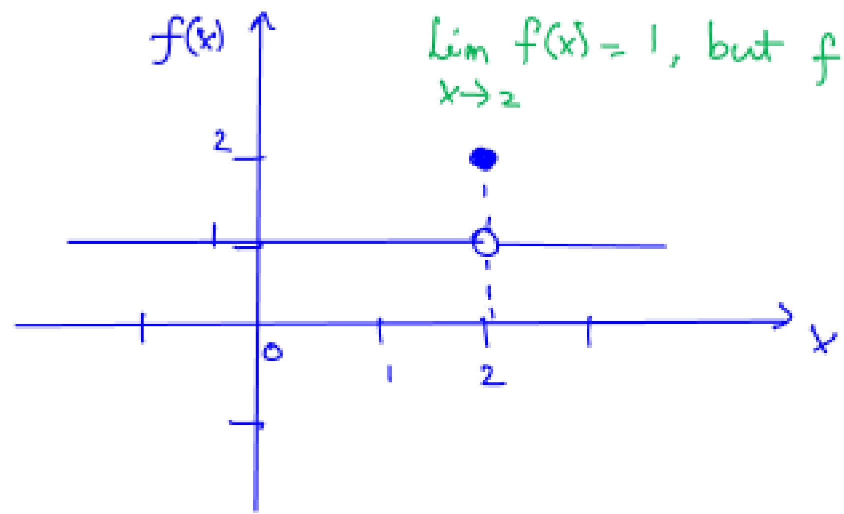


(b) $\lim_{x \to 2} f(x)$ exists, and f is discontinuous at $x=2$

$\lim_{x \to 2} f(x) = 1$, but $f(2) = 2$

Here is one such f

$$f(x) = \begin{cases} 1 & x \neq 2 \\ 2 & x = 2. \end{cases}$$



(c) $\lim_{x\to 2^-} f(x) = f(2)$, and f is discontinuous at $x=2$.

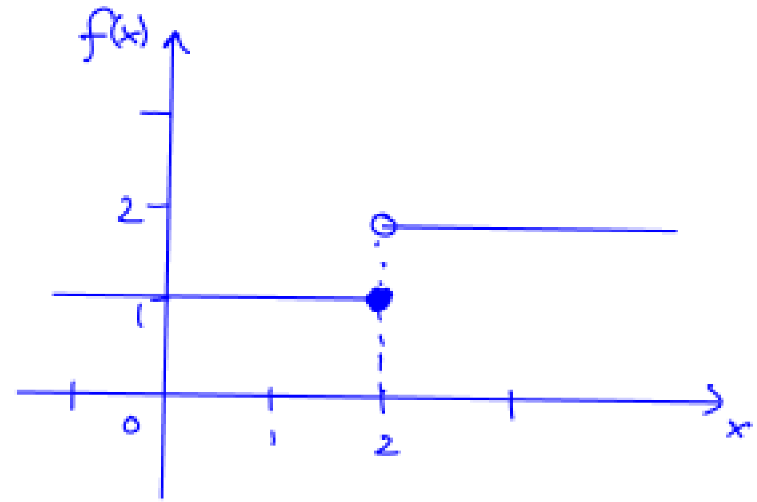
Here is one such f

$$f(x) = \begin{cases} 1 & x \leq 2 \\ 2 & x > 2. \end{cases}$$

here,

$\lim_{x\to 2^-} f(x) = f(2) = 1$;

however, $\lim_{x\to 2^+} f(x) = 2$.



#4 Determine the equation of the tangent line to the curve $y = -3x^2 + 5x - 2$ at the point $(1,0)$.

We begin by finding the slope of the tangent line to the curve at $x=1$.

Let $f(x) = -3x^2 + 5x - 2$. Then, slope $m = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

Therefore, $m = \lim_{x\to 1} \dfrac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \dfrac{(-3x^2+5x-2)-0}{x-1} = \lim_{x\to 1} \dfrac{\cancel{(x-1)}(-3x+2)}{\cancel{(x-1)}} = -1.$

Now, using the point-slope equation of a line with slope m passing through (x_1, y_1)

$$y - y_1 = m(x - x_1), \text{ or}$$

$$y - 0 = (-1)(x - 1), \text{ or}$$

$$\boxed{y = -x + 1.}$$

#5

Function	Derivative
a	iii
b	i
c	ii