HOMEWORK 2 - KEY

#1 In the following exercises, decide whether the geometric series converges or diverges. Justify your answer. If the series converges, compute its sum.

(a) $\frac{5}{4} + \frac{5}{8} + \frac{5}{16} + \frac{5}{32} + \cdots\cdots$

First term, $a = 5/4$ Common ratio, $r = 1/2$

Since $|r| < 1$, the series converges.

Sum, $S = \frac{a}{1-r} = \frac{5/4}{1-1/2} = \frac{5/4}{1/2} = \boxed{\frac{5}{2}}$

(b) $1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots + \left(-\frac{3}{2}\right)^{n-1} + \cdots\cdots$

First term, $a = 1$ Common ratio, $r = -3/2$

Since $|r| \geq 1$, the series diverges

(c) $\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \cdots\cdots$

First term, $a = \frac{1}{2}$ Common ratio, $r = -1$

Since $|r| \geq 1$, the series diverges.

(d) $-\frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} + \cdots\cdots$

First term, $a = -\frac{1}{4}$ Common ratio, $r = -\frac{1}{4}$

Since $|r| < 1$, the series converges.

Sum, $S = \frac{a}{1-r} = \frac{-\frac{1}{4}}{1-(-\frac{1}{4})} = \frac{-\frac{1}{4}}{\frac{5}{4}} = \boxed{-\frac{1}{5}}.$

#2 Compute the following sums without using a calculator.

(a) $S = 198 + 194 + 190 + \cdots + 66$

The terms form an arithmetic sequence with first term $a_1 = 198$ and common difference, $d = -4$

To find the number of terms in the series, we use

$$a_n = a_1 + (n-1)d$$

$\Rightarrow \quad 66 = 198 + (n-1)(-4)$

$\Rightarrow \quad 4(n-1) = 198 - 66$

$\Rightarrow \quad 4(n-1) = 132$

$\Rightarrow \quad n-1 = 33$

$\Rightarrow \quad \underline{n = 34}$

Therefore, $S = \frac{n}{2}(a_1 + a_n)$
(partial sum)

$$= \frac{34}{2}(198 + 66)$$

$$\boxed{S = 17(264)} = 4488$$

(b) $S = 6561 - 2187 + 729 + \cdots - 3$

This is a geometric series with first term $\underline{a_1 = 6561}$ and common ratio, $\underline{r = \frac{-2187}{6561} = -\frac{1}{3}}$.

To find the number of terms, n, we use

$$a_n = a_1 r^{n-1}$$

$\Rightarrow \quad -3 = 6561 \left(-\frac{1}{3}\right)^{n-1}$

$\Rightarrow \quad (-3)\left(-\frac{1}{3}\right) = 6561 \left(-\frac{1}{3}\right)^{n-1}\left(-\frac{1}{3}\right)$

$$\Rightarrow 1 = 6561\left(\frac{-1}{3}\right)^n$$

$$\Rightarrow \frac{1}{6561} = \left(\frac{-1}{3}\right)^n$$

$$\Rightarrow \left(-\frac{1}{3}\right)^8 = \left(\frac{-1}{3}\right)^n$$

$$\Rightarrow \underline{n = 8.}$$

Therefore, $S = S_n = \dfrac{a_1(1-r^n)}{1-r} = \dfrac{6561\left(1-\left(\frac{-1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)}$

$$= \frac{6561\left(1-\frac{1}{6561}\right)}{1+\frac{1}{3}}$$

$$= \frac{\cancel{6561}\left(\frac{6560}{\cancel{6561}}\right)}{\frac{4}{3}}$$

$$= \frac{6560}{4/3}$$

$$\boxed{S = 3(1640) = 4920}$$

#3 If $\{y_n\}_n$ is a geometric sequence, compute the sum

$$\frac{\sqrt{y_2}}{\sqrt{y_2}-\sqrt{y_1}} + \frac{\sqrt{y_3}}{\sqrt{y_3}-\sqrt{y_2}} + \cdots + \frac{\sqrt{y_{n+1}}}{\sqrt{y_{n+1}}-\sqrt{y_n}}$$

in terms of n and the common ratio r. All terms in the sum are well defined.

Since $\{y_n\}_n$ is a geometric sequence, we have $y_2 = y_1 r$ where r is the common ratio. Therefore, we may rewrite the first term as

$$\frac{\sqrt{y_2}}{\sqrt{y_2}-\sqrt{y_1}} = \frac{\sqrt{y_1 r}}{\sqrt{y_1 r}-\sqrt{y_1}} = \frac{\cancel{\sqrt{y_1}}\ \sqrt{r}}{\cancel{\sqrt{y_1}}\,(\sqrt{r}-1)} = \frac{\sqrt{r}}{\sqrt{r}-1}.$$

Similarly, $$\frac{\sqrt{y_3}}{\sqrt{y_3}-\sqrt{y_2}} = \frac{\sqrt{y_2 r}}{\sqrt{y_2 r}-\sqrt{y_2}} = \frac{\cancel{\sqrt{y_2}}\ \sqrt{r}}{\cancel{\sqrt{y_2}}\,(\sqrt{r}-1)} = \frac{\sqrt{r}}{\sqrt{r}-1},$$ and

$$\frac{\sqrt{y_{n+1}}}{\sqrt{y_{n+1}}-\sqrt{y_n}} = \frac{\sqrt{y_n r}}{\sqrt{y_n r}-\sqrt{y_n}} = \frac{\cancel{\sqrt{y_n}}\ \sqrt{r}}{\cancel{\sqrt{y_n}}\,(\sqrt{r}-1)} = \frac{\sqrt{r}}{\sqrt{r}-1}.$$

Hence, $$\frac{\sqrt{y_2}}{\sqrt{y_2}-\sqrt{y_1}} + \frac{\sqrt{y_3}}{\sqrt{y_3}-\sqrt{y_2}} + \cdots + \frac{\sqrt{y_{n+1}}}{\sqrt{y_{n+1}}-\sqrt{y_n}} = \underbrace{\frac{\sqrt{r}}{\sqrt{r}-1} + \frac{\sqrt{r}}{\sqrt{r}-1} + \cdots + \frac{\sqrt{r}}{\sqrt{r}-1}}_{n \text{ terms}}$$

$$\boxed{\frac{\sqrt{y_2}}{\sqrt{y_2}-\sqrt{y_1}} + \frac{\sqrt{y_3}}{\sqrt{y_3}-\sqrt{y_2}} + \cdots + \frac{\sqrt{y_{n+1}}}{\sqrt{y_{n+1}}-\sqrt{y_n}} = \frac{n\sqrt{r}}{\sqrt{r}-1}.}$$

#4 Compute the limit $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\cdots+\frac{1}{3^n}}$$

The numerator is a geometric series with first term $a_1 = 1$ and common ratio $r = \frac{1}{2}$.

We have $$S_n = 1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n} = \frac{a_1(1-r^n)}{1-r} = \frac{1\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}} = 2\left(1-\tfrac{1}{2^n}\right).$$

The denominator is a geometric series with $b_1 = 1$ and common ratio $\tilde{r} = \frac{1}{3}$.

We have $$S_n = 1 + \tfrac{1}{3} + \cdots + \tfrac{1}{3^n} = \frac{b_1(1-\tilde{r}^n)}{1-\tilde{r}} = \frac{1\left(1-\frac{1}{3^n}\right)}{1-\frac{1}{3}} = \frac{3}{2}\left(1-\tfrac{1}{3^n}\right)$$

$$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\cdots+\frac{1}{3^n}} = \lim_{n\to\infty} \frac{2\left(1-\frac{1}{2^n}\right)}{\frac{3}{2}\left(1-\frac{1}{3^n}\right)} = \frac{\lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right)}{\lim_{n\to\infty} \frac{3}{2}\left(1-\frac{1}{3^n}\right)}$$

$$= \frac{2-0}{3/2-0}$$

$$\boxed{\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{2^n}}{1+\frac{1}{3}+\cdots+\frac{1}{3^n}} = \frac{4}{3}}$$

#5

(a) $S_1 = \text{area}(\Delta ABC) = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(6)(4) = 12$ sq. units

this area is less than shaded area in figure 1.

(b) $\text{area}(\Delta ACD_1) = \text{area}(\Delta BCD_2) = \frac{1}{2}(5)(0.6) = 1.5$ sq. units

obtained by Pythogorean rule

$$\frac{\text{area}(\Delta ACD_1)}{\text{area}(\Delta ABC)} = \frac{1.5}{12} = \frac{1}{8}$$

$S_2 = 12 + 2(1.5) = 12 + 3 = 15$ sq. units.

(c) $S_3 = 12 + 2(1.5) + 4\left(\frac{1}{8} \times 1.5\right)$

$= 12 + 3 + \frac{3}{4}$

$S_3 = 15.75$ sq. units.

(d) area, $S = 12 + 2\left(\frac{1}{8} \times 12\right) + 4\left(\frac{1}{8} \times 1.5\right) + 8\left(\frac{1}{8} \times \frac{3}{16}\right) + \cdots\cdots$ sq. units

$= 12 + 3 + \frac{3}{4} + \cdots\cdots + \frac{12}{4^{n-1}} + \cdots\cdots$

this is a geometric series with first term $a = 12$ and common ratio $r = \frac{1}{4}$.

Since $|r| < 1$, this series converges with

$$S = \frac{a}{1-r} = \frac{12}{1 - 1/4} = \frac{12}{3/4} = 16 \text{ sq. units}$$