MTH 305 – HOMEWORK 1 (KEY)

#1 (a) first term, $x_1 = 3 - 2\left(\frac{4}{3}\right) = \frac{1}{3}$

Direct formula: $x_n = \frac{1}{3} + \frac{4}{3}(n-1)$, $n \geq 1$.

Recursive formula: $x_1 = \frac{1}{3}$, $x_{n+1} = x_n + \frac{4}{3}$, $n \geq 1$.

(b) first term, $x_1 = \frac{1}{\left(\frac{3}{4}\right)} = \frac{4}{3}$

Direct formula: $x_n = \frac{4}{3}\left(\frac{3}{4}\right)^{n-1}$ for $n \geq 1$.

Recursive formula: $x_1 = \frac{4}{3}$, $x_{n+1} = \frac{3}{4} x_n$, $n \geq 1$.

(c) We know for an arithmetic sequence $x_n = x_1 + (n-1)d$

Hence
$$x_8 = 40 = x_1 + 7d$$
$$x_{20} = -20 = x_1 + 19d$$

Solving these two equations, we obtain $d = -5$ and $x_1 = 75$.

Therefore, $x_n = 75 - 5(n-1)$, $n \geq 1$ (direct formula)

$x_1 = 75$, $x_{n+1} = x_n - 5$, $n \geq 1$. (recursive formula)

(d) For a geometric sequence, $x_n = x_1 r^{n-1}$. Hence,

$$x_5 = 10 = x_1 r^4 \quad \text{—— ①}$$
$$x_8 = -10 = x_1 r^7 \quad \text{—— ②}$$

Dividing ② by ①, we get $r^3 = -1$ or $r = -1$.

Substituting in ①, we get $x_1 = 10$.

Therefore, (direct formula) $x_n = 10(-1)^{n-1}$, $n \geq 1$

(recursive formula) $x_1 = 10$, $x_{n+1} = -x_n$, $n \geq 1$.

#2

(a) this is a geometric sequence with $x_1 = 2$ and $r = -\frac{1}{3}$.

(direct formula) $x_n = 2\left(\frac{-1}{3}\right)^{n-1}$ for $n \geq 1$.

(recursive formula) $x_1 = 2$, $x_{n+1} = -\frac{x_n}{3}$ for $n \geq 1$.

(b) this is an arithmetic sequence with $x_1 = 5$ and $d = -3$

(direct formula) $x_n = 5 - 3(n-1)$, $n \geq 1$.

(recursive formula) $x_1 = 5$, $x_{n+1} = x_n - 3$, $n \geq 1$.

(c) this is an arithmetic sequence with $x_1 = \frac{4}{3}$ and $d = \frac{1}{3}$

(direct formula) $x_n = \frac{4}{3} + \frac{1}{3}(n-1)$, $n \geq 1$.

(recursive formula) $x_1 = \frac{4}{3}$, $x_{n+1} = x_n + \frac{1}{3}$, $n \geq 1$.

#3 (a) this sequence is convergent with $L = 1$.

Why? as n increases $\left(-\frac{1}{2}\right)^n$ gets smaller and $\left(-\frac{1}{2}\right)^n \to 0$ as $n \to \infty$. Therefore $x_n \to 1$ as $n \to \infty$.

Hence $L = \lim_{n\to\infty} x_n = 1$

(Alternatively, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = 1 + \lim_{n\to\infty} \left(-\frac{1}{2}\right)^n$.

applying the sandwich theorem to the second term, $-\frac{1}{n} \leq \left(-\frac{1}{2}\right)^n \leq \frac{1}{n}$ and hence,

$\lim_{n\to\infty} \left(-\frac{1}{2}\right)^n = \lim_{n\to\infty} \frac{1}{n} = 0$. the result follows.)

(b) this sequence is convergent with $L = 0$.

this is a geometric sequence with common ratio $|r| < 1$ (here $r = 0.7$).

therefore as $n \to \infty$, $r^n \to 0$; consequently $x_n \to 0$.

(note: we can also use the sandwich theorem here since $0 \leq x_n \leq \frac{7}{n}$)

(c) this sequence is divergent.

Suppose there exists an L such that $L = \lim_{n\to\infty} x_n$. We consider 3 cases:

case ① ($L = 0$) Consider the interval $(-1, 1)$. Since $x_n \notin (-1, 1)$ for all $n \geq 1$, there are infinitely many terms not contained in this interval. Hence the sequence cannot converge to 0.

Case 2 $(L>0)$. Consider the interval $(0, 2L)$. Again, there are infinitely many terms

this is an open interval centred at L

of the sequence which are not contained in this interval since $x_n \notin (0, 2L)$ for all $n > L$. Hence the sequence cannot converge to any positive real number.

Case 3 $(L<0)$. Consider the interval $(2L, 0)$. Again, there are infinitely many terms

this is an open interval centred at L

of the sequence which are not contained in this interval since $x_n \notin (2L, 0)$ for all $n \in \mathbb{N}$. Hence the sequence cannot converge to any negative real number.

From all 3 cases, we conclude that there is no real number L with $L = \lim_{n\to\infty} x_n$.

Therefore the sequence is divergent.

(d) Factoring out n^3, we have $a_n = -\dfrac{3+n}{n^3+2} = -\dfrac{\cancel{n^3}\left(\frac{3}{n^3}+\frac{1}{n^2}\right)}{\cancel{n^3}\left(1+\frac{2}{n^3}\right)}$

$$\lim_{n\to\infty} a_n = -\frac{\lim_{n\to\infty}\left(\frac{3}{n^3}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(1+\frac{2}{n^3}\right)} = -\frac{\lim_{n\to\infty}\frac{3}{n^2} + \lim_{n\to\infty}\frac{1}{n^2}}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}\frac{2}{n^3}} = -\frac{0+0}{1+0} = 0.$$

the sequence is convergent with $L=0$.

(e) $a_n = \dfrac{1+2+\cdots+n}{n^2} = \dfrac{\frac{n(n+1)}{2}}{n^2} = \dfrac{n^2+n}{2n^2} = \dfrac{\cancel{n^2}(1+1/n)}{\cancel{n^2}\cdot 2}$

Now, $\lim\limits_{n\to\infty} a_n = \dfrac{\lim\limits_{n\to\infty}(1+1/n)}{\lim\limits_{n\to\infty} 2} = \dfrac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}}{2} = \dfrac{1+0}{2} = \dfrac{1}{2}.$

the series is <u>convergent with $L = 1/2$.</u>

#4

(a) $0, 1, 2, 5, 12, 29, \ldots$ sequence $\{p_n\}_{n\in\mathbb{N}}$

(b) $2, 2, 6, 14, 34, 82, \ldots$ Sequence $\{q_n\}_{n\in\mathbb{N}}$.

(c) $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$ sequence $\{x_n\}_{n\in\mathbb{N}}$

(or $1, 1.5, 1.4, 1.41666, 1.41379, \ldots$)

Guess is $L = \lim\limits_{n\to\infty} x_n = \sqrt{2}$.