Formula Sheet – Final Exam – MTH 305 (Spring 2017)

(1) **Arithmetic Sequence**
A sequence given by a recursive formula of the form $x_{n+1} = x_n + d, n \geq 1$ is called an *arithmetic sequence* with first term x_1 and common difference d. The direct formula for such a sequence is $x_n = x_1 + (n-1)d, n \geq 1$.
The sum of the first n consecutive terms of an arithmetic sequence is

$$s_n = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} [x_1 + (i-1)d] = \frac{n}{2}(x_1 + x_n) = nx_1 + \frac{n(n-1)}{2}d.$$

(2) **Geometric Sequence**
A sequence given by a recursive formula of the form $y_{n+1} = y_n \cdot r, n \geq 1$ is called a *geometric sequence* with first term y_1 and common ratio r. The direct formula for such a sequence is $y_n = y_1 \cdot r^{n-1}, n \geq 1$.
The sum of the first n consecutive terms of a geometric sequence is

$$s_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \left[y_1 \cdot r^{i-1}\right] = \begin{cases} y_1 \frac{r^n - 1}{r-1} & \text{if} \quad r \neq 1 \\ ny_1 & \text{if} \quad r = 1. \end{cases}$$

(3) **Convergence of Geometric Series**
If a is a nonzero real number, the series $a + ar + ar^2 + \ldots$ *converges* whenever $|r| < 1$, in which case $\displaystyle\sum_{i=1}^{\infty} a \cdot r^{i-1} = \frac{a}{1-r}$ and *diverges* whenever $|r| \geq 1$.

(4) **Properties of Limits of Sequences**
Let $\{x_n\}_n$, $\{y_n\}_n$ be two *convergent* sequences and let c be a real number. Then the following hold:
(a) $\lim\limits_{n\to\infty} (x_n + y_n) = \lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} y_n$
(b) $\lim\limits_{n\to\infty} (x_n - y_n) = \lim\limits_{n\to\infty} x_n - \lim\limits_{n\to\infty} y_n$
(c) $\lim\limits_{n\to\infty} (c \cdot y_n) = c \cdot \lim\limits_{n\to\infty} y_n$
(d) $\lim\limits_{n\to\infty} (x_n \cdot y_n) = \lim\limits_{n\to\infty} x_n \cdot \lim\limits_{n\to\infty} y_n$
(e) $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = \dfrac{\lim\limits_{n\to\infty} x_n}{\lim\limits_{n\to\infty} y_n}$ (provided $y_n \neq 0$ for all n and $\lim\limits_{n\to\infty} y_n \neq 0$)

(5) **Sandwich theorem for sequences**
If the sequences $\{x_n\}_n$, $\{y_n\}_n$, and $\{z_n\}_n$ are such that
(a) $x_n \leq y_n \leq z_n$ for each n, and
(b) $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = L$ for some $L \in \mathbb{R}$,
then the sequence $\{y_n\}_n$ is convergent, and $\lim\limits_{n\to\infty} y_n = L$.

(6) **Function**
A function is determined by two sets, A, B, and a law, f (correspondence, assignment, rule) that associates to *each* element x in A a *unique* element y in B. The set A is called the *domain* of the function, while B is called the *codomain* of the function. In addition, $\text{Range}(f) = \{f(x) : x \in A\}$.

(7) **Composite Functions**
Suppose f and g are real-valued functions. Then the *composition* of f with g, denoted $f \circ g$, is the function $(f \circ g)(x) = f(g(x))$, which is defined for all real values x in the domain of g *and* for which $g(x)$ is in the domain of f.

(8) **Limit of a function**

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

(9) **Limit Laws**
Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

(a) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
(b) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
(c) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
(d) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
(e) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (provided $\lim_{x\to a} g(x) \neq 0$)
(f) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer
(g) $\lim_{x\to a} c = c$
(h) $\lim_{x\to a} x = a$
(i) $\lim_{x\to a} x^n = a^n$ where n is a positive integer
(j) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. (If n is even, we assume that $a > 0$.)
(k) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer.
(If n is even, we assume that $\lim_{x\to a} f(x) > 0$.)

(10) **Direct substitution property**
If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

(11) **Squeeze/Sandwich theorem**
If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

(12) **Continuity**
A function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

(a) Any polynomial is continuous everywhere (i.e., it is continuous on $\mathbb{R}$)
(b) Any rational (as well as root and trigonometric) function is continuous wherever it is defined (i.e., it is continuous on its domain)

(13) **Intermediate Value Theorem**
Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

(14) **Exponent Rules**
Fix a real number $b > 0$ and take m, n to be whole numbers. Then

(a) $b^m = b \cdot b \cdots b$ (m copies of b)
(b) $b^{-m} = \frac{1}{b^m}$
(c) $b^{\frac{1}{n}} = \sqrt[n]{b}$
(d) $b^{\frac{m}{n}} = \sqrt[n]{b^m}$
(e) $b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$

More generally, fix a real number $b > 0$ and let x be a real number. Then $f(x) = b^x$ is an exponential function with base b. For all real numbers x, y, we have

(a) $b^{-x} = \frac{1}{b^x}$
(b) $b^{x+y} = b^x b^y$
(c) $(b^x)^y = b^{xy}$

(15) **Rates of Change and Derivatives**

The average rate of change of a function f over an interval $[a, b]$ is by definition the value of

$$\frac{f(b) - f(a)}{b - a}.$$

The derivative of a function f at point x, denoted by $f'(x)$, (also called the *instantaneous rate of change of* f *at* x) is the limit

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

provided it exists. We say f is differentiable at x. Alternatively,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

If f is differentiable at x, then $f'(x)$ equals the slope of the tangent line to the graph of f at the point $(x, f(x))$.

(16) **Position, Velocity, Acceleration**

If $h(t)$ denotes the position function at time t of an object moving along a straight line, then $v(t) = h'(t)$ is the velocity of the object at time t, $s(t) = |v(t)|$ is the object's speed at time t, while $a(t) = v'(t) = h''(t)$ is the object's acceleration at time t.

(17) **Relation between f and f'**

If f is a differentiable function on (a, b), then the following are true.

(a) f is nonincreasing on (a, b) if and only if $f' \leq 0$ on (a, b).
(b) f is nondecreasing on (a, b) if and only if $f' \geq 0$ on (a, b).
(c) If $f' < 0$ on (a, b), then f is decreasing on (a, b).
(d) If $f' > 0$ on (a, b), then f is increasing on (a, b).
(e) The tangent line to the graph of f at $(c, f(c))$ is horizontal if and only if $f'(c) = 0$.

(18) **Derivative Rules**

Here the functions f and g are supposed to be differentiable, the functions appearing in the denominators are assumed to be nonzero, and f is assumed to be positive when considering $(\sqrt{f})'$.

(a) Power rule: $(x^a)' = ax^{a-1}$ for any rational number a.
(b) Constant multiple rule: $(cf)' = c(f)'$ for any real number c.
(c) Sum and difference rule: $(f \pm g)' = f' \pm g'$
(d) Product rule: $(f \cdot g)' = f' \cdot g + f \cdot g'$
(e) Reciprocal rule: $\left(\frac{1}{f}\right)' = -\frac{f'}{f^2}$
(f) Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$
(g) Square-root rule: $(\sqrt{f})' = \frac{f'}{2\sqrt{f}}$
(h) Chain rule: $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ (at points x where both $f(g(x))$ and $f'(g(x))$ are defined).

(19) **Derivatives of Some Common Functions**

(a) $\frac{d}{dx}(e^x) = e^x$
(b) $\frac{d}{dx}(a^x) = a^x \ln a$ (Note: $\ln x$ is the same as $\log_e x$)
(c) $\frac{d}{dx}(\ln x) = \frac{1}{x}$
(d) $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
(e) $\frac{d}{dx}(\sin x) = \cos x$
(f) $\frac{d}{dx}(\cos x) = -\sin x$
(g) $\frac{d}{dx}(\tan x) = \sec^2 x$

(20) **Antiderivatives**
The set of all antiderivatives of f is denoted by $\int f(x)dx$; for simplicity, we write $\int f(x)dx = F(x)+C$, where F is an arbitrary antiderivative of f and C is a numerical constant.

(21) **Properties of (Indefinite) Integrals**
- (a) $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$ (Power rule)
- (b) $\int \frac{1}{x}\,dx = \ln|x| + C$ (Logarithmic rule)
- (c) $\int e^x\,dx = e^x + C$ (Exponential rule)
- (d) $\int kf(x)\,dx = k\int f(x)\,dx$ (Scalar multiplication rule)
- (e) $\int(f \pm g)(x)\,dx = \int f(x)\,dx \pm \int g(x\,dx$ (Addition/Difference rule)
- (f) $\int u^n(x)u'(x)\,dx = \frac{1}{n+1}u^{n+1}(x) + C, n \neq -1$ (General power rule)
- (g) $\int e^{u(x)}u'(x)\,dx = e^{u(x)} + C$ (General exponential rule)
- (h) $\int f[g(x)]g'(x)\,dx = f[g(x)] + C$ (Chain rule)

(22) **Definite Integrals**
The definite integral of f from a to b, denoted by $\int_a^b f(x)\,dx$, is by definition the variation of any antiderivative of F of f over the interval $[a,b]$; that is, $F(b) - F(a)$.

(23) **Properties of Definite Integrals**
- (a) $\int_a^a f(x)\,dx = 0$
- (b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- (c) $\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$
- (d) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
- (e) $\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

(24) **Method of Substitution**
To evaluate an (indefinite) integral $\int f(x)\,dx$ using the method of substitution,
- (a) Choose a new variable $u = u(x)$ that would simplify integration.
- (b) Compute $u'(x)$.
- (c) Express the integral in terms of u by using the fact that $du = u'(x)dx$.
- (d) Evaluate the integral using this new expression.
- (e) Rewrite your final answer in terms of x.

(25) **Integration by Parts**
$$\int uv'\,dx = uv - \int u'v\,dx.$$