Formula Sheet – Mid-Term Exam – MTH 305 (Spring 2017)

(1) **Arithmetic Sequence**
A sequence given by a recursive formula of the form $x_{n+1} = x_n + d, n \geq 1$ is called an *arithmetic sequence* with first term x_1 and common difference d. The direct formula for such a sequence is $x_n = x_1 + (n-1)d, n \geq 1$.
The sum of the first n consecutive terms of an arithmetic sequence is

$$s_n = \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} [x_1 + (i-1)d] = \frac{n}{2}(x_1 + x_n) = nx_1 + \frac{n(n-1)}{2}d.$$

(2) **Geometric Sequence**
A sequence given by a recursive formula of the form $y_{n+1} = y_n \cdot r, n \geq 1$ is called a *geometric sequence* with first term y_1 and common ratio r. The direct formula for such a sequence is $y_n = y_1 \cdot r^{n-1}, n \geq 1$.
The sum of the first n consecutive terms of a geometric sequence is

$$s_n = \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} \left[y_1 \cdot r^{i-1}\right] = \begin{cases} y_1 \frac{r^n - 1}{r-1} & \text{if} \quad r \neq 1 \\ ny_1 & \text{if} \quad r = 1. \end{cases}$$

(3) **Convergence of Geometric Series**
If a is a nonzero real number, the series $a + ar + ar^2 + \ldots$ *converges* whenever $|r| < 1$, in which case $\sum_{i=1}^{\infty} a \cdot r^{i-1} = \frac{a}{1-r}$ and *diverges* whenever $|r| \geq 1$.

(4) **Properties of Limits of Sequences**
Let $\{x_n\}_n$, $\{y_n\}_n$ be two *convergent* sequences and let c be a real number. Then the following hold:
(a) $\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$
(b) $\lim_{n\to\infty} (x_n - y_n) = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n$
(c) $\lim_{n\to\infty} (c \cdot y_n) = c \cdot \lim_{n\to\infty} y_n$
(d) $\lim_{n\to\infty} (x_n \cdot y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n$
(e) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$ (provided $y_n \neq 0$ for all n and $\lim_{n\to\infty} y_n \neq 0$)

(5) **Sandwich theorem for sequences**
If the sequences $\{x_n\}_n$, $\{y_n\}_n$, and $\{z_n\}_n$ are such that
(a) $x_n \leq y_n \leq z_n$ for each n, and
(b) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$ for some $L \in \mathbb{R}$,
then the sequence $\{y_n\}_n$ is convergent, and $\lim_{n\to\infty} y_n = L$.

(6) **Function**
A function is determined by two sets, A, B, and a law, f (correspondence, assignment, rule) that associates to *each* element x in A a *unique* element y in B. The set A is called the *domain* of the function, while B is called the *codomain* of the function. In addition, $\text{Range}(f) = \{f(x) : x \in A\}$.

(7) **Composite Functions**
Suppose f and g are real-valued functions. Then the *composition* of f with g, denoted $f \circ g$, is the function $(f \circ g)(x) = f(g(x))$, which is defined for all real values x in the domain of g *and* for which $g(x)$ is in the domain of f.

(8) **Limit of a function**

$$\lim_{x\to a} f(x) = L \qquad \text{if and only if} \qquad \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

(9) **Limit Laws**

Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

(a) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(c) $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$

(d) $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$

(e) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ (provided $\lim_{x\to a} g(x) \neq 0$)

(f) $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

(g) $\lim_{x\to a} c = c$

(h) $\lim_{x\to a} x = a$

(i) $\lim_{x\to a} x^n = a^n$ where n is a positive integer

(j) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer. (If n is even, we assume that $a > 0$.)

(k) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer.
(If n is even, we assume that $\lim_{x\to a} f(x) > 0$.)

(10) **Direct substitution property**

If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

(11) **Squeeze/Sandwich theorem**

If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

(12) **Continuity**

A function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

(a) Any polynomial is continuous everywhere (i.e., it is continuous on $\mathbb{R}$)

(b) Any rational (as well as root and trigonometric) function is continuous wherever it is defined (i.e., it is continuous on its domain)

(13) **Intermediate Value Theorem**

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.