Lab 2: We are Rocket Scientists

This lab focuses on (1) exploring the growth properties of the *natural logarithm* and *natural exponential* functions introduced in sections 6.2 and 6.3 of Stewart and (2) illustrating one of the principal philosophies for solving real world problems using calculus. The philosophy is, in a nutshell, that *it can be useful to approximate discrete processes with continuous ones* because continuous processes can be analyzed using the methods of calculus, while discrete ones cannot.

Point (2) and its reverse (solving continuous problems using discrete approximations; which describes basically all computer simulations) are themes that we will revisit often throughout this course. They are also what makes calculus such a powerful tool in engineering disciplines.

1.1 - What's the point?

In the previous lab we've explored the (in)feasibility of space elevators, we now turn our attention to the more conventional way of getting satellites up into space. As mentioned in the first lab, rocket launches are "very expensive, dangerous, and consume incredible amounts of fuel"; the first two problems are invariably tied to the third.

If you look at the modern Atlas V rocket, out of its 335-ton (metric) weight, only between 4 to 20 tons are the payload, an efficiency of about 6%. The fuel required for the lanuch takes up about 300 tons, or about 90% of the total launch weight. Why does it need so much fuel?

To understand this, first we need to understand how rockets work.

1.2 - How do rockets work? A crash course in physics

Unlike airplanes which work by complicated laws of fluid dynamics, rockets can be entirely explained by Newton's third law: for every action that is an equal but opposite reaction. The exhaust spits out one way, and the rocket accelerates in the other direction. To understand this a bit better, let us play some "fantasy baseball" with help from Robin Roberts, who happens to be the only MSU alum in the baseball Hall of Fame.

A baseball weighs 5 to 5.25 ounces (approximate 145 grams or 0.145 kg). Robin can throw at 90 mph (approximately 40 m/s). Using that

$$\text{Momentum} = \text{Mass} \cdot \text{Velocity}$$

the momentum carried by a major league fastball is about $40 \times 0.145 = 5.8$ kg m/s.

Now the fantasy part: suppose the baseball diamond's surface is replaced by ice. What happens to Robin after he pitches? The quantitative description of Newton's third law is called *conservation of momentum*. Initially Robin and the baseball are at rest, so they have zero momentum. After the pitch, the ball flies one way with 5.8 kg m/s of momentum, which by the conservation law means that Robin must slide in the opposite direction with also 5.8 kg m/s of momentum. Now Robin is listed at 190 lbs (about 86 kg), so dividing the momentum by his mass we can find his speed $5.8/86 \approx 0.067$ m/s (or about 2.5 inches per second; slow but noticeable).

2.1 - The Robin Roberts Rocket

Now let us build a Rocket out of Robin Roberts and a giant bag of baseballs.

The equation that governs the Robin-Roberts-Rocket is the conservation of momentum

$$\text{Momentum before a pitch} = \text{Momentum after the pitch}$$

The momentum before the pitch is

$$\text{Momentum before the pitch} = (M_{RR} + k \cdot M_{BB}) \times v_i \tag{1}$$

where $M_{RR} = 86$ kg is the mass of Robin Roberts, $M_{BB} = 0.145$ kg is the mass of one baseball, and k is the number of baseballs in the bag before the pitch. The velocity v_i is the speed at which Robin, together with his bag of balls, is moving before his pitch.

The momentum after the pitch is

$$\text{Momentum after pitch} = (M_{RR} + (k-1) \cdot M_{BB}) \times v_f + M_{BB} \times (v_i - v_B) \tag{2}$$

where $v_B = 40$ m/s is the speed at which Robin throws his baseball.

DISCUSSION QUESTION 1

Answer discussion question 1 on your lab answer sheet now.

2.2 - Simulation

We can make this in code form.

In [ ]:
```
% First define the constants
M_RR = 86; % Mass of Robin Roberts
M_BB = 0.145; % Mass of one baseball
v_B = 40; % Speed of Robin's fastball

% Next define a function that, when given the number of baseballs initia
% computes the final velocity (v_f in equation (2) above).
v_f = @(v_i, k) v_i + (M_BB * v_B) / (M_RR + (k-1) * M_BB)
```

Remember the syntax for MATLAB where to define a function depending on several inputs, we write

```
function = @(input1, input2, ..., inputN) expression
```

Now let's wrap this code in a loop to see how much speed our Robin-Roberts-Rocket can pick up, after throwing all his balls. (For more about the `while` loop, see the documentation (http://www.mathworks.com/help/matlab/ref/while.html?refresh=true).)

In [ ]:
```
num_balls = 500; %Initial number of balls
speed = 0; %Very initial velocity, in m/s

while num_balls > 0
    speed = v_f(speed,num_balls);
    num_balls = num_balls - 1;
end

speed  % Show the final speed
```

RESULTS QUESTION 1

Answer results question 1 on your lab answer sheet now.

DISCUSSION QUESTION 2

Answer discussion question 2 on your lab answer sheet now.

2.3 - Melon Husk's request

The billionaire Melon Husk came by and wants to know if our Robin Roberts Rocket can begin from a standing start, accelerate to a specified speed, and slow down again close to a stop. To simulate this, we need to modify our code slightly.

The following code outputs the maximum speed attained, as well as the final speed after Robin Roberts has exhausted all the baseballs. The input data are the number of balls allotted for the accelerating phase of the rocket, and the number of balls allotted for the decelerating phase of the rocket.

In [ ]:

```
% We define a new function to compute the final velocity based on the in
% of balls k, when Robin Roberts is now throwing the ball in the opposit
v_f2 = @(v_i, k)  v_i - (M_BB * v_B) / (M_RR + (k-1) * M_BB);

% Data
num_balls_accelerating = 500; % number of balls to use for phase 1, the 
num_balls_decelerating = 500; % number of balls to use for phase 2, the 
speed = 0; % Initial velocity, in m/s

while num_balls_accelerating > 0
    speed = v_f(speed,num_balls_accelerating+num_balls_decelerating);
    num_balls_accelerating = num_balls_accelerating - 1;
end

maxspeed = speed;

while num_balls_decelerating > 0
    speed = v_f2(speed,num_balls_decelerating);
    num_balls_decelerating = num_balls_decelerating - 1;
end

maxspeed, speed
```

RESULTS QUESTION 2

Answer results question 2 on your lab answer sheet now.

DISCUSSION QUESTION 3

Answer discussion question 3 on your lab answer sheet now.

3 - The Rocket Equation

Try running the code from Section 2.3 with the number of balls used in phase 1 being 15 million, and the number of balls used in phase 2 being 100,000. That is the approximate number of balls needed to get to the maximum speed of 200 m/s, which is roughly the speed of an airliner. Notice how long it took the code to run! For real-life simulations (where the "baseballs" are replaced by "tiny molecules of exhaust") it will be extremely impractical to compute the amount of fuel needed by trial-and-error using code like the above.

And here, calculus comes to the rescue.

3.1 - Derivation of the rocket equation

Let M denote the total mass on board our rocket: $M = M_{RR} + k \times M_{BB}$ when there are k baseballs left; then while accelerating, we can think of the speed v of our rocket as a function of the mass M. Then rearranging your answer to Discussion Question 1, part 2 (or alternatively the code in Section 2.2) we get that

$$\frac{v_f - v_i}{M_{BB}} = \frac{v_B}{M - M_{BB}}$$

Now, since Robin Roberts weigh a lot more than a baseball, we can pretend that M_{BB} is infinitesimal. Similarly, the change of speed given by one baseball (0.067 m/s from our discussion in Section 2) is very small, and we can also pretend it to be an infinitesimal quantity.

This leads us to the following equation

$$\frac{\mathrm{d}v}{\mathrm{d}M} = -\frac{v_B}{M} \tag{3}$$

Here, $v(M)$ is the function describing the velocity of the rocket ad a function of the remaining mass onboard the rocket. The number v_B is a constant, which describes the speed of the rocket exhaust. The minus sign in front of the right hand side comes from the fact that the the mass of the rocket is *decreasing*.

Equation (3) is called the *ideal rocket equation*. We can integrate it using what we've learned about natural logarithms! This gives

$$v(M_f) - v(M_i) = v_b \left(\ln(M_f) - \ln(M_i)\right) \tag{4}$$

The great thing about equation (4) is that we can solve this for the mass in terms of the desired velocity. In the equation, the velocity $v(M_f)$ is the final velocity at the and of the accleration process, $v(M_i)$ is the inital velocity before accelerating. M_i and M_f are the total mass of the rocket before and after acceleration respectively.

RESULTS QUESTION 3

Answer results question 3 on your lab answer sheet now.

3.2 - Bonus material: now add gravity

So far our Robin Roberts Rocket has just been zooming around on top of an ice rink. But real rockets (for going into space) need to go *up*. So we have to add the effects of gravity into the equation. Going back to equation (3); instead of writing v as a function of M, we can write both v and M as function of t, the time. Then by chain rule we have that

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}M}\frac{\mathrm{d}M}{\mathrm{d}t}$$

so in the absence of gravity equation (3) reads

$$Mv' + v_B M' = 0$$

where v_B again is the speed of the exhaust, taken to be a constant.

If we were to add gravity, the force of gravity introduces a gravitational acceleration of $g = 9.8$ m/s^2 (for everything up to low-earth orbit, the decrease in the gravitational acceleration due to the increase in altitude is negligible, being no more than 10% of the total). So the equation becomes

$$M(t)v'(t) + v_B M'(t) + M(t)g = 0 \tag{5}$$

Notice that $M'(t) < 0$ as the rocket is losing mass to its exhaust.

Some Data

Chemical propellants (such as those used in the Atlas V (https://en.wikipedia.org/wiki/Atlas_V)) can provide a $v_B \approx 4000$ m/s.

Engineering limits of controlled burn (instead of explosions) allow the rocket fuel to be consumed at a maximum of about 1000 kg/s (for a single rocket engine). (In other words, if we don't want our rocket to blow up, the rate of fuel consumption $M'(t)$ has to be, in units of kg/s, $-1000 < M'(t) < 0$.)

To achieve lift-off, the rocket must have $v'(0) > 0$. So rearranging equation (5) we have

$$v'(0) = \frac{-v_B M'(0)}{M(0)} - g = \frac{4000000}{M(0)} - 9.8 \tag{6}$$

So our engineering limits put an upper limit of maximum lift-off-weight (this includes the payload, the rocket itself, and the fuel!) for a single engine rocket to be approximately

$$M(0) = \frac{4000000}{9.8} \approx 400000 \text{ kilograms}$$

The Atlas-V's launch weight of 335 tons is pretty close to the engineering limit.

Simulation

The following code simulates the vertical launch of a chemical rocket with maximal burn rate of 1000 kg/s and exhaust velocity of 4000 m/s.

In [ ]:
```
% Modifiable parameters
Vehicule_Mass = 8000; % The mass of the rocket including all structure c
     % and the fuel tank. Measured in kg.
Propellent_Mass = 150000; % The mass of fuel carried on board at launch.
     % We will calculate the fuel-tank size as proportional to the prope
     % the fuel tank needs to weigh around 10% of the propellent
Fuel_Tank_Ratio = 0.1; % Mass of Fuel Tank / Propellent mass

% As the sum of the Vehicule_Mass and the Propellent_Mass (multiplied by
% is the total mass of the rocket initially, try to make sure that the s
% you don't get absurd answers.

%%% Code
height = [];
velocity = [];
height(1) = 1; % Suppose initially the rocket is still on a 1 meter high
velocity(1) = 0; % Initially at rest
Burn_Rate = 1000; % The rate of fuel consumption, in kg/s.


current_mass = Vehicule_Mass + Propellent_Mass * (1 + Fuel_Tank_Ratio);

timing_resolution = 0.01; % Time-step-size in seconds; keep it less than

while height(end) > 0
    if current_mass <= Vehicule_Mass + Fuel_Tank_Ratio * Propellent_Mass
        velocity(end+1) = velocity(end) - timing_resolution * 9.8 / (1 +
    else
        velocity(end+1) = velocity(end) - timing_resolution * 9.8 / (1 +
        current_mass = current_mass - Burn_Rate * timing_resolution;
    end

    height(end+1) = height(end) + velocity(end) * timing_resolution;
end
```

The code above stores the flight history of our rocket in terms of two arrays. `height` and `velocity`. We can plot the height as a function of time like below. We see that for the default values (150 metric tons of fuel launching an 8000 kilogram object) we easily reach "outer space" (which is 10^5 meters from the surface of the earth), and quite easily get into the height of the so-called "medium earth orbit" (1200 to 22000 miles).

In [ ]:
```
plot(height)
xlabel({'elapsed time', ['(in units of ', num2str(timing_resolution), '
ylabel({'height', ['(in units of meters)']})
```

But as you can see, in our simulation, the rocket eventually comes back down. Part of it is because we launched it straight up (so no sideways motion to allow it to orbit). But does it have, in principle, enough speed to get into orbit? The following plot shows two curves, one is the speed of the rocket, the other is the equivalent orbital velocity at the corresponding height.

In [ ]:
```
plot(velocity(1:150000))   % Plot only the first 1500 seconds of flight
hold on
plot(7900 ./ sqrt(1 + height(1:150000) ./ 6400000 ))
legend('Speed of rocket','Orbital velocity at corresponding height', 'Lo
xlabel({'elapsed time',['(in units of ',num2str(timing_resolution),' sec
ylabel('speed (m/s)')
```

A `Vehicule_Mass` of 8000 kilograms corresponds roughly to the launch of a DirectTV satellite. Playing around with the graphs above, you see that with modern technology, it is in fact not possible to launch such a satellite into orbit (the vehicule speed never exceeds the orbital velocity!) using a single-stage, single-engine rocket. The modern solution is to use "multi-stage" rockets. The basic idea is this:

1. For single stage rockets, the fuel tank remains attached to the launch vehicule at all times.
2. For multi-stage rockets, the fuel tank is segmented, and one segment empties, the fuel-tank separates and drops off.

Think about why it is that the multi-stage rockets can achieve a higher velocity for the same amount of fuel.

An alternative possibility is to reduce the amount of material used for the fuel tank. Current structural engineering limitations means that the tank must at minimum weigh 8% of the fuel that it contains. Play around with the simulation above by modifying the `Fuel_Tank_Ratio` parameter to see how, with improvements in material science and structural engineering that it may become possible to launch a DirectTV satellite using a single stage rocket and only 150 ton of fuel.

In [ ]:
```
```