Name: ______________________________

READ THE FOLLOWING INSTRUCTIONS.

- **Do not open your exam until told to do so.**
- No calculators, cell phones or any other electronic devices can be used on this exam.
- Clear your desk of everything except pens, pencils and erasers.
- Without fully opening the exam, check that you have pages 1 through 20.
- Fill in your name, etc. on this first page.
- **Show all your work.** Write your answers clearly! Include enough steps for the grader to be able to follow your work. Include words to clarify your reasoning.
- Do first all of the problems you know how to do immediately. Do not spend too much time on any particular problem. Return to difficult problems later.
- If you have any questions please raise your hand and a proctor will come to you.
- There is no talking allowed during the exam.
- You will be given exactly 80 minutes for this exam.

ACADEMIC HONOR CODE

As a student and citizen of the Michigan State University Community I pledge to not lie, cheat, or steal in my academic endeavors.

I have read and understand the above instructions: ______________________________

SIGNATURE

Quick Answer Questions. No partial credit available; No justification necessary.

1. Fill in the blanks below.

(a) (2 points) Determine if the following sequences are bounded.

(i) $\left\{(-1)^n(2n^2-3)-27\right\}_{n\in\mathbb{N}}$ NOT BOUNDED.

(ii) $\left\{\dfrac{3n}{n+8}\right\}_{n\in\mathbb{N}}$ BOUNDED.

(b) (4 points) Determine if the given equation describes y as a function of x.

(i) $x=-2y^2+6$ NO

(ii) $y^2=7x^2-6$ NO

(iii) $y=\sqrt{x-16}$ YES

(iv) $2y^5-7x=8$ YES

(c) (4 points)

$$\text{Let}\qquad g(x)=\begin{cases} x & \text{if } x<1\\ 3 & \text{if } x=1\\ 2-x^2 & \text{if } 1<x\le 2\\ x-3 & \text{if } x>2\end{cases}$$

Evaluate each of the following, if it exists.

1. $\lim\limits_{x\to 1^-} g(x)$ 1
2. $\lim\limits_{x\to 1} g(x)$ 1
3. $\lim\limits_{x\to 2^+} g(x)$ -1
4. $\lim\limits_{x\to 2} g(x)$ DOES NOT EXIST

Additional Work Space:

Complete Explanation Questions. Provide complete justifications for your responses.

2. Determine direct and recursive formulas for the given sequences below. Give the value of the sequence's first term.

   (a) (5 points) An arithmetic sequence $\{x_n\}_{n\geq 1}$ with $x_3 = 0$ and $x_7 = 40$.

(n^{th} term) $x_n = x_1 + (n-1)d$

Hence,

for $n=3$, $0 = x_1 + 2d$ ——— (1)

for $n=7$, $40 = x_1 + 6d$ ——— (2)

(2) − (1) gives, $40 = 4d$, or $\boxed{d=10}$

Substituting in (1), $x_1 = -2d = -2(10)$, or $\boxed{x_1 = -20}$

Therefore, (direct formula) $x_n = -20 + 10(n-1)$, $n \geq 1$

(recursive formula) $x_{n+1} = x_n + 10$, $n \geq 1$

$x_1 = -20$

   (b) (5 points) A geometric sequence $\{y_n\}_{n\geq 1}$ with $y_3 = -6$ and $y_6 = \frac{2}{9}$.

(n^{th} term) $y_n = y_1 r^{n-1}$, $n \geq 1$

Hence,

for $n=3$, $-6 = y_1 r^2$ ——— (a)

$n=6$, $\frac{2}{9} = y_1 r^5$ ——— (b)

(b) ÷ (a) give, $\frac{2/9}{-6} = r^3$, or, $-\frac{1}{27} = r^3$, or $\boxed{r = -\frac{1}{3}}$

Substituting in (a), $-6 = y_1 \left(\frac{-1}{3}\right)^2$, or $\boxed{y_1 = -54}$

Therefore, (direct formula) $y_n = -54\left(-\frac{1}{3}\right)^{n-1}$, $n \geq 1$

(recursive formula) $y_{n+1} = -\frac{1}{3} y_n$, $n \geq 1$

$y_1 = -54$

Additional Work Space:

3. For each of the following, determine whether or not they converge. If they converge, what is the limit? Provide some algebraic justification.

(a) (5 points) $\left\{\dfrac{(-1)^n}{n^2}+\dfrac{3}{4}\right\}_{n\in\mathbb{N}}$

$\boxed{\text{CONVERGES}}$

as $n\to\infty$, $\frac{(-1)^n}{n^2}\to 0$. More formally, using the sandwich theorem,

$-\frac{1}{n} \leq \frac{(-1)^n}{n^2} \leq \frac{1}{n}$ for all $n\in\mathbb{N}$.

Since $\lim\limits_{n\to\infty}\left(-\frac{1}{n}\right) = \lim\limits_{n\to\infty}\left(\frac{1}{n}\right) = 0$, it follows that

$\lim\limits_{n\to\infty}\frac{(-1)^n}{n^2} = 0$.

By limit laws, it follows that $\lim\limits_{n\to\infty}\left(\frac{(-1)^n}{n^2}+\frac{3}{4}\right) = \lim\limits_{n\to\infty}\frac{(-1)^n}{n^2} + \lim\limits_{n\to\infty}\frac{3}{4}$

$L = 0 + \frac{3}{4}$

$\boxed{L = \frac{3}{4}}$

(b) (6 points) $\left\{n-\sqrt{n^2+7}\right\}_{n\in\mathbb{N}}$

$\boxed{\text{CONVERGES}}$

Rewriting the n^{th} term, $x_n = n-\sqrt{n^2+7} = n-\sqrt{(n^2+7)} \cdot \frac{n+\sqrt{(n^2+7)}}{n+\sqrt{(n^2+7)}}$

$$= \frac{n^2-\left(\sqrt{n^2+7}\right)^2}{n+\sqrt{n^2+7}}$$

$$= \frac{\cancel{n^2}-(\cancel{n^2}+7)}{n+\sqrt{n^2+7}}$$

$$= \frac{-7}{n+\sqrt{n^2+7}}$$

as n increases, the denominator gets larger and x_n gets smaller

we have $\lim\limits_{n\to\infty}\left(n-\sqrt{n^2+7}\right) = 0$.

Additional Work Space:

4. (8 points) Using the properties of limits, find the limit of the following convergent sequence

$$\left\{\frac{4n}{n+7}\right\}_{n\in\mathbb{N}}.$$

$$\lim_{n\to\infty}\left(\frac{4n}{n+7}\right) = \lim_{n\to\infty}\left[\frac{\cancel{n}\cdot 4}{\cancel{n}\cdot(1+7/n)}\right]$$ (factoring out n from the denominator)

$$= \lim_{n\to\infty} \frac{4}{1+7/n}$$

$$= \frac{\lim_{n\to\infty} 4}{\lim_{n\to\infty}(1+7/n)}$$ (by property 4(e))

$$= \frac{\lim_{n\to\infty} 4}{\lim_{n\to\infty} 1 + \lim_{n\to\infty}(7/n)}$$ (by property 4(a))

$$= \frac{\lim_{n\to\infty} 4}{\lim_{n\to\infty} 1 + 7\cdot\lim_{n\to\infty}\left(\frac{1}{n}\right)}$$ (by property 4(c))

$$= \frac{4}{1+7(0)}$$ (by using $\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$)

or; $$\boxed{\lim_{n\to\infty}\left(\frac{4n}{n+7}\right) = 4}$$

5. Decide whether the geometric series below converge or diverge. Justify your answer. If the series converges, compute its sum.

(a) (5 points) $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27} + \dots$

$\boxed{\text{CONVERGES}}$

This is a geometric series with first term $\underline{a = 2}$ and common ratio $\underline{r = -\frac{1}{3}}$. Since $|r| = \frac{1}{3} < 1$, the series converges.

Sum $L = \frac{a}{1-r} = \frac{2}{1-(-\frac{1}{3})} = \frac{2}{1+\frac{1}{3}} = \frac{2}{4/3}$

or $\boxed{L = \frac{3}{2}.}$

(b) (5 points) $\sum_{n=1}^{\infty} \frac{3}{7^{n-1}}$

Rewriting, we have
$$\sum_{n=1}^{\infty} \frac{3}{7^{n-1}} = 3 \sum_{n=1}^{\infty} \frac{1}{7^{n-1}}$$
$$= 3 \sum_{n=1}^{\infty} \frac{7}{7^n}$$
$$= 3(7) \sum_{n=1}^{\infty} \frac{1}{7^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{7^n}$ is a geometric series with first term $\underline{a = \frac{1}{7}}$ and common ratio $\underline{r = \frac{1}{7}}$, it converges (since $|r| = \frac{1}{7} < 1$) with sum
$$L = \frac{a}{1-r} = \frac{1/7}{1 - 1/7} = \frac{1}{6}.$$

Therefore, the series $\sum_{n=1}^{\infty} \frac{3}{7^{n-1}}$ $\boxed{\text{Converges}}$ with sum $L' = 3(7)\left(\frac{1}{6}\right)$

$\boxed{\text{Sum} = \frac{7}{2}}$

Additional Work Space:

6. (8 points) Compute the sum $S = -\frac{1}{4} + \frac{1}{2} - 1 + \cdots + 128$ without using a calculator.

the terms in the above sum for a geometric sequence with first term $\underline{a = -\frac{1}{4}}$ and common ratio $\underline{r = -2}$.

to find number of terms in the sum

we have (n^{th} term) $a_n = a_1 r^{n-1}$

hence, $128 = \left(-\frac{1}{4}\right)(-2)^{n-1}$

$\Rightarrow 128(-4) = (-2)^{n-1}$

$\Rightarrow -512 = (-2)^{n-1}$

Since $(-2)^9 = -512$, we get $\boxed{n = 10}$

Now, we have $S_{10} = \dfrac{a(1-r^n)}{1-r}$

$= \dfrac{(-1/4)(1-(-2)^{10})}{1-(-2)}$

$= \dfrac{(-1/4)(1-2^{10})}{3}$

$= \left(-\frac{1}{12}\right)(1-1024)$

$= \left(-\frac{1}{12}\right)(-1023)$

$$\boxed{S = S_{10} = \frac{1023}{12}}$$

7. (8 points) Let $x = 0.\overline{75}$. Show that $x = \frac{75}{99}$. (note: the notation $0.\overline{75}$ means that the digits 7 and 5 repeat indefinitely; i.e., $x = 0.757575757575\ldots$.

We have

$$x = 0.\overline{75}$$
$$= \frac{75}{100} + \frac{75}{10000} + \frac{75}{10^6} + \cdots$$

This is a geometric series with first term $\underline{a = 75/100}$ and common ratio $\underline{r = \frac{1}{100}}$.

Since $|r| = \frac{1}{100} < 1$, this series converges with sum

$$x = \frac{a}{1-r}$$
$$= \frac{75/100}{1 - 1/100}$$
$$= \frac{75/100}{99/100}$$

or $\boxed{x = \frac{75}{99}}$

8. (10 points) Find the assignment rule (or law/formula) and the domain of the specified composite functions below:

$$f \circ g \quad \text{and} \quad g \circ f, \quad \text{if} \quad f(x) = 1 - \frac{1}{x} \quad \text{and} \quad g(x) = \frac{x}{x+3}.$$

Note:

domain$(f) = \{x \in \mathbb{R} : x \neq 0\}$

domain$(g) = \{x \in \mathbb{R} : x \neq -3\}$

$\boxed{f \circ g}$

$$(f\circ g)(x) = f(g(x))$$
$$= f\left(\frac{x}{x+3}\right)$$
$$= 1 - \frac{1}{\left(x/x+3\right)}$$
$$\boxed{(f\circ g)(x) = 1 - \left(\frac{x+3}{x}\right) = \frac{-3}{x}}$$

Domain

- want $x \in$ domain(g)
  $\Rightarrow x \neq -3$
  AND
- want $g(x) \in$ domain(f)
  $\Rightarrow \frac{x}{x+3} \neq 0$
  $\Rightarrow x \neq 0$

$$\boxed{\text{domain}(f\circ g) = \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq -3\}}$$

$\boxed{g \circ f}$

$$(g\circ f)(x) = g(f(x))$$
$$= g\left(1 - \frac{1}{x}\right)$$
$$= \frac{(1 - 1/x)}{(1-1/x)+3}$$
$$= \frac{x-1}{x-1+3x}$$

$$\boxed{\begin{array}{l} (g\circ f)(x) = \dfrac{x-1}{4x-1} \\ \text{domain}(g\circ f) = \{x \in \mathbb{R} : x \neq 0 \text{ and } x \neq 1/4\} \end{array}}$$

Domain

- want $x \in$ domain(f)
  $\Rightarrow x \neq 0$
  AND
- want $f(x) \in$ domain(g)
  $\Rightarrow 1 - \frac{1}{x} \neq -3$
  $\Rightarrow 4 \neq \frac{1}{x}$
  $\Rightarrow x \neq \frac{1}{4}$

Additional Work Space:

9. Evaluate the following limits, if they exist. Show the steps used and use limit properties to justify your answer.

(a) (5 points) $\lim_{x\to 5} \frac{x^2 - x - 20}{x - 5}$

$$\lim_{x\to 5} \frac{x^2-x-20}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x+4)}{\cancel{(x-5)}}$$ (factorizing the numerator)

$$= \lim_{x\to 5} (x+4)$$

$$= \lim_{x\to 5} x + \lim_{x\to 5} 4$$ (by property 9(a))

$$= 5 + 4$$ (by properties 9(g), 9(h))

$$\lim_{x\to 5} \frac{x^2-x-20}{x-5} = 9$$

(b) (6 points) $\lim_{x\to -1} \frac{\sqrt{x^2+3} - 2}{x+1}$

$$\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x+1} = \lim_{x\to -1}\left(\frac{\sqrt{x^2+3}-2}{x+1}\right)\left(\frac{\sqrt{x^2+3}+2}{\sqrt{x^2+3}+2}\right)$$ (multiply by conjugate)

$$= \lim_{x\to -1} \frac{(\sqrt{x^2+3})^2 - 2^2}{(x+1)(\sqrt{x^2+3}+2)}$$ (using $(a+b)(a-b) = a^2-b^2$)

$$= \lim_{x\to -1} \frac{(x^2+3)-4}{(x+1)(\sqrt{x^2+3}+2)}$$

$$= \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+3}+2)}$$

$$= \lim_{x\to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{(x+1)}(\sqrt{x^2+3}+2)}$$ (using $(a+b)(a-b) = a^2-b^2$)

$$= \frac{\lim_{x\to -1} (x-1)}{\lim_{x\to -1} (\sqrt{x^2+3}+2)}$$ (by property 9(e))

Additional Work Space:

$$\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x+1} = \frac{\lim_{x\to -1} x - \lim_{x\to -1} 1}{\lim_{x\to -1}\sqrt{x^2+3} + \lim_{x\to -1} 2}$$ (by properties 9(a), 9(b))

$$= \frac{\lim_{x\to -1} x - \lim_{x\to -1} 1}{\sqrt{\lim_{x\to -1}(x^2+3)} + \lim_{x\to -1} 2}$$ (by property 9(k))

$$= \frac{\lim_{x\to -1} x - \lim_{x\to -1} 1}{\sqrt{\lim_{x\to -1} x^2 + \lim_{x\to -1} 3} + \lim_{x\to -1} 2}$$ (by property 9(a))

$$= \frac{-1-1}{\sqrt{(-1)^2+3}+2}$$ (by property 9(g), 9(h))

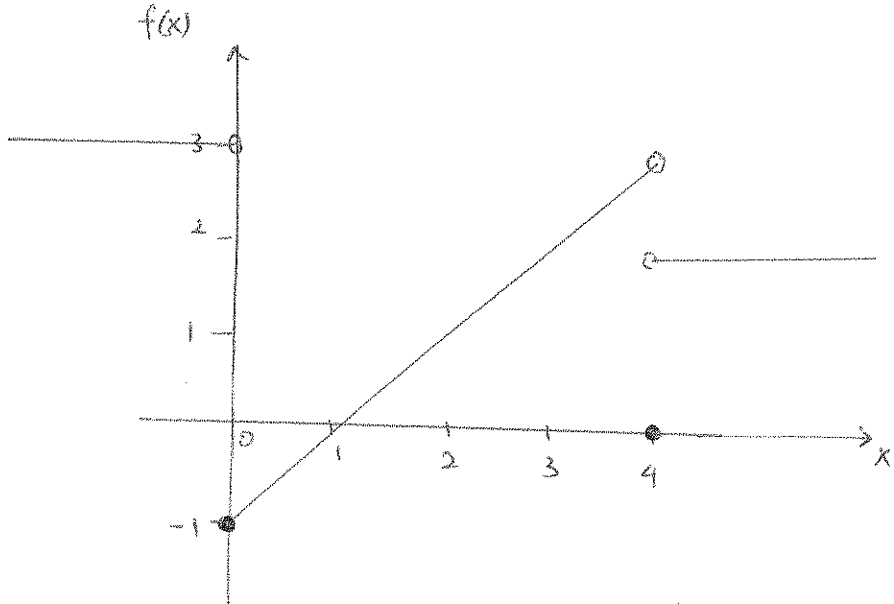
$$= \frac{-2}{2+2}$$

$$\lim_{x\to -1} \frac{\sqrt{x^2+3}-2}{x+1} = -\frac{1}{2}$$

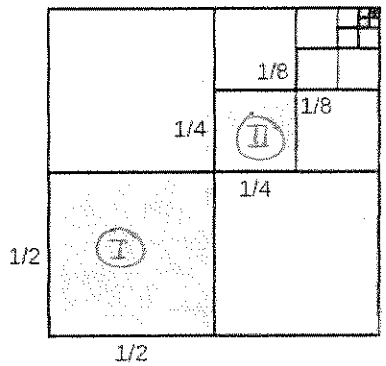
10. (6 points) Sketch the graph of an example of a function f that satisfies all of the given conditions.

$$\lim_{x\to 0^-} f(x) = 3, \quad \lim_{x\to 0^+} f(x) = -1, \quad \lim_{x\to 4^-} f(x) = 3, \quad \lim_{x\to 4^+} f(x) = 2 \quad f(0) = -1, \quad f(4) = 0.$$

there are (infinitely) many possible choices. Here is one such f:



11. (8 points) Find the sum of the areas of the shaded squares in the figure below. (Note: you may assume that there are an infinite number of shaded squares following the same pattern.)



We have,

$$\text{area of the shaded squares} = \underbrace{\left(\tfrac{1}{2}\right)^2}_{\substack{\text{area of}\\ \text{largest square}\\ \text{(denoted I)}}} + \underbrace{\left(\tfrac{1}{4}\right)^2}_{\substack{\text{area of}\\ \text{next largest}\\ \text{square (denoted II)}}} + \left(\tfrac{1}{8}\right)^2 + \cdots\cdots$$

$$= \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots\cdots$$

this is a geometric series with first term $a = \frac{1}{4}$ and common ratio $r = \frac{1}{4}$

Since $|r| = \frac{1}{4} < 1$, the series converges with

$$(\text{area}) \text{ sum} = \frac{a}{1-r} = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3}$$

Therefore sum of the areas of the shaded squares is $\boxed{S = \frac{1}{3}.}$

Additional Work Space:

Congratulations you are now done with the exam!
Go back and check your solutions for accuracy and clarity.

When you are completely happy with your work please bring your exam to the front to be handed in.
Please have your MSU student ID ready so that is can be checked.

DO NOT WRITE BELOW THIS LINE.

Page	Points	Score
2	10	
4	10	
6	11	
8	8	
9	10	
11	8	
12	8	
13	10	
15	11	
17	6	
18	8	
Total:	100	