CONTINUITY

Def^n: A function f is continuous at a number a if

$$\lim_{x\to a} f(a) = f(a)$$

Note: this requires 3 conditions

* $f(a)$ is defined (or, a is in domain(f))
* $\lim_{x\to a} f(a)$ exists $\left(\text{i.e., } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)\right)$
* $\lim_{x\to a} f(x) = f(a)$

Geometrically, if the graph of a function has no break in it, then the function is continuous.

$\boxed{\text{Def}^n:}$ A function f is continuous on an interval if it is continuous at every number in the interval.

$\boxed{\text{Def}^n}$ A function f is continuous from the right at a number a if $\lim_{x \to a^+} f(x) = f(a)$.

$\boxed{\text{Theorem}}$

* A polynomial is continuous everywhere
* A rational function is continuous wherever it is defined; ie, it is continuous on its domain.

Example:

① On what intervals is each of the following functions continuous?

(a) $f(x) = \sqrt{x}$ (b) $g(x) = \dfrac{x^2+3x+17}{x^2-4}$ (c) $h(x) = \sqrt{x} - \dfrac{x+2}{x-2} + \dfrac{x-1}{x^2+9}$

(a) $f(x) = \sqrt{x}$

f is continuous on $[0, \infty)$

(b) g is a rational function; hence, it is continuous on its domain.

Note that domain of g is $\mathbb{R} \setminus \{2, -2\}$

or, $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$

(c) $h(x) = \underbrace{\sqrt{x}}_{F(x)} - \underbrace{\dfrac{x+2}{x-2}}_{G(x)} + \underbrace{\dfrac{x-1}{x^2+9}}_{H(x)}$

F is continuous on $[0, \infty)$

G — " — on $\mathbb{R} \setminus \{2\}$

H — " — for all real values.

h is continuous on $[0, 2) \cup (2, \infty)$

② Let $f(x) = \begin{cases} 3 - x^2 & \text{if } x \leq 0 \\ (x-1)^2 + a & \text{if } x > 0. \end{cases}$

Find the value of a such that $f(x)$ is continuous at $x = 0$.

We have,

$$f(0) = 3 - 0^2 = 3.$$

Also,

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (3 - x^2) = 3 - 0 = 3.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x-1)^2 + a = (-1)^2 + a = 1 + a.$$

For continuity at $x = 0$, we require $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

Therefore, we require $1 + a = 3$

$$\Rightarrow \boxed{a = 2}$$