PROPERTIES

Operations on limits. Some general combination rules make most limit computations routine. Suppose we know that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then we have the Limit Laws:

- *Sum:* $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
- *Difference:* $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$.
- *Constant Multiple:* $\lim_{x\to a}(c\,f(x)) = c\,\lim_{x\to a} f(x)$, for a constant c.
- *Product:* $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$.
- *Quotient:* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
- *Power:* $\lim_{x\to a} f(x)^n = (\lim_{x\to a} f(x))^n$, for a whole number n.
- *Root:* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, for a whole number* n.

These all have the form: "The limit of an operation equals the operation applied to the limits." These Laws are also valid for one-sided limits.

THEOREM

$\lim_{x\to a} f(x) = L$ if and only if $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Example

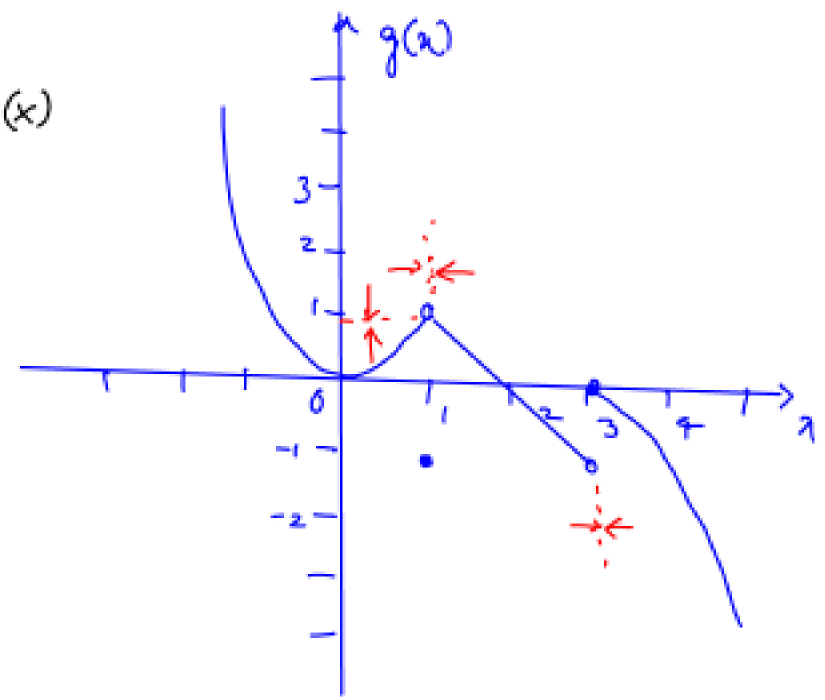
Let $g(x) = \begin{cases} x^2 & \text{if } x < 1 \\ -1 & \text{if } x = 1 \\ -x+2 & \text{if } 1 < x < 3 \\ -x^2+6x-9 & \text{if } x \geq 3. \end{cases}$

(a) Evaluate each of the following, if it exists

(i) $\lim_{x \to 1^-} g(x)$ (ii) $\lim_{x \to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x \to 3^-} g(x)$ (v) $\lim_{x \to 3^+} g(x)$ (vi) $\lim_{x \to 3} g(x)$

(b) Sketch the graph of g.



(a) (i) $\boxed{\lim_{x \to 1^-} g(x)}$

For $x < 1$, we see that $g(x) = x^2$.

Hence, $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} x^2 = 1$.

(ii) $\boxed{\lim_{x\to 1} g(x)}$ We know from (a), $\lim_{x\to 1^-} g(x) = 1$.

Moreover, since $g(x) = -x+2$ when $1 < x < 3$, we have

$$\lim_{x\to 1^+} g(x) = \lim_{x\to 1^+} (-x+2) = -1+2 = 1.$$

Since $\lim_{x\to 1^-} g(x) = \lim_{x\to 1^+} g(x) = 1$, we can conclude that $\lim_{x\to 1} g(x) = 1$.

(iii) $g(1) = -1$ (from function definition)

(iv) $\boxed{\lim_{x\to 3^-} g(x)}$ For $1 < x < 3$, we have $g(x) = -x+2$.

Hence $\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-} (-x+2) = -3+2 = -1$.

(v) $\boxed{\lim_{x\to 3^+} g(x)}$ we have $g(x) = -x^2+6x-9$ for $x \geq 3$.

Therefore $\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+} (-x^2+6x-9) = -(3^2)+6(3)-9 = 0$.

(vi) $\boxed{\lim_{x\to 3} g(x)}$ does not exist since $\lim_{x\to 3^-} g(x) \neq \lim_{x\to 3^+} g(x)$

"Infinite Limits"

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Using Limit properties to evaluate limits

Evaluate $\lim_{x \to -2} \frac{x^3 + 3x^2 + 1}{4 - 3x}$

$$\lim_{x \to -2} \frac{x^3 + 3x^2 + 1}{4 - 3x} = \frac{\lim_{x \to -2} (x^3 + 3x^2 + 1)}{\lim_{x \to -2} (4 - 3x)}$$

(Quotient rule)

$$= \frac{\lim_{x \to -2} x^3 + 3 \lim_{x \to -2} x^2 + \lim_{x \to -2} 1}{\lim_{x \to -2} 4 - 3 \lim_{x \to -2} x}$$

(properties on sum/difference, scalar multiplication)

$$= \frac{(-2)^3 + 3((-2)^2) + 1}{4 - 3(-2)} = \frac{-8 + 12 + 1}{4 + 6}$$

$$= \frac{1}{2}$$